The Schreier Property and Gauss' Lemma

D. D. Anderson and M. Zafrullah

Abstract

Let D be an integral domain with quotient field K. Recall that D is Schreier if D is integrally closed and for all $x, y, z \in D \backslash \{0\}$, $x|yz$ implies that $x = rs$ where $r|y$ and $s|z$. A GCD domain is Schreier. We show that an integral domain D is a GCD domain if and only if (i) for each pair $a, b \in D \backslash \{0\}$, there is a finitely generated ideal B such that $aD \cap bD = B_v$ and (ii) every quadratic in $D[X]$ that is a product of two linear polynomials in $K[X]$ is a product of two linear polynomials in $D[X]$. We also show that D is Schreier if and only if every polynomial in $D[X]$ with a linear factor in $K[X]$ has a linear factor in $D[X]$ and show that D is a Schreier domain with algebraically closed quotient field if and only if every nonconstant polynomial over D is expressible as a product of linear polynomials. We also compare the two most common modes of generalizing GCD domains. One is via properties that imply Gauss' Lemma and the other is via the Schreier property. The Schreier property is not implied by any of the specializations of Gauss' Lemma while all but one of the specializations of Gauss Lemma are implied by the Schreier property.

2000 Mathematics Subject Classification: 13A05 and 13F99.
Keywords and phrases: GCD domain, Schreirer domain, Gauss' Lemma.

1 Introduction and basics

Let D be an integral domain with quotient field K. Call D *pre-Schreier* if for all $x, y, z \in D^* = D \backslash \{0\}$, $x|yz$ implies that $x = rs$ where $r|y$ and $s|z$. An integrally closed pre-Schreier domain is called a *Schreier domain*. Schreier domains were introduced by Cohn [9] where it was shown that a GCD domain is a Schreier domain. In this note we show that an integral domain D is a GCD domain if and only if (i) for each pair $a, b \in D^*$, there is a finitely generated ideal B such that $aD \cap bD = B_v = (B^{-1})^{-1}$ and (ii) every quadratic in $D[X]$ that is a product of two linear polynomials in $K[X]$ is a product of two linear polynomials in $D[X]$. We also show that D is Schreier if and only if every polynomial in $D[X]$ that has a linear factor in $K[X]$ has a linear factor in $D[X]$ and show that D is a Schreier domain with algebraically closed quotient field if and only if every nonconstant polynomial over D is expressible as a product of linear polynomials. We give non-Bezout examples of Schreier domains with algebraically closed quotient fields. We also compare the two most common modes of generalizing GCD domains. Of these one is via properties that imply Gauss' Lemma and the other is that a GCD domain is Schreier which in turn is pre-Schreier. It appears that Cohn's Schreier property is not implied by any of the specializations of Gauss' Lemma while all but one of the specializations of Gauss Lemma are implied by the Schreier property. Yet, both approaches when applied to atomic domains produce UFD's. We also show how to avoid using Gauss' Lemma via Nagata type theorems.

To keep the article self contained we include a brief description of the v-operation and related notions. For more information see Gilmer [12, Sections 32 and 34].

Let D be an integral domain with quotient field K. A fractional ideal F of D is a D-submodule of K such that dF is an ideal of D for some $d \in D^*$. Let $F(D)$ be the set of nonzero fractional ideals of D. For each $A \in F(D)$ define $A^{-1} = D :_K A = \{x \in K | xA \subset D\}$. Denote $(A^{-1})^{-1}$ by A_v. The association $A \mapsto A_v$ is a function on $F(D)$, called the *v-operation*, that has the following easy to establish properties: for $a \in K^*$ and $B, C \in F(D)$

(1) $(aD)_v = aD,\ (aA)_v = aA_v$,

(2) $A \subseteq A_v$ and $A \subseteq B$ implies $A_v \subseteq B_v$, and

(3) $(A_v)_v = A_v$.

In addition to these, the v-operation can be shown to satisfy the following properties:

(4) $(A_v)^{-1} = A^{-1}$,

(5) if $A_i \in F(D)$ and if $\sum A_i \in F(D)$, $(\sum A_i)_v = (\sum (A_i)_v)_v$,

(6) $(AB)_v = (A_v B)_v = (A_v B_v)_v$ (usually called v-multiplication), and

(7) if $A_i \in F(D)$ and $\cap A_i \in F(D)$ then $(\cap (A_i)_v)_v = \cap (A_i)_v$.

Call $A \in F(D)$ a *v-ideal* if $A = A_v$ and call A a *v-ideal of finite type* if $A = B_v$ for a finitely generated fractional ideal B. Combining (1) and (7) a nonzero intersection of principal fractional ideals is a v-ideal.

In Section 2 we show that D is a GCD domain if and only if every quadratic polynomial $f(X)$ over D that splits as a product of two linear polynomials in $K[X]$ also splits as a product of two linear polynomials over $D[X]$ and $aD \cap bD$ is a v-ideal of finite type for every pair $a, b \in D^*$. Recall that an integral domain D is called a *GCD domain* if $GCD(a, b)$ exists for each pair $a, b \in D^*$. Equivalently, D is a GCD domain if and only if $LCM(a, b)$ exists for each pair $a, b \in D^*$ if and only if $aD \cap bD$ is principal for each pair $a, b \in D^*$ if and only if $(a, b)_v$ is principal for all $a, b \in D^*$. Using the last characterization and the properties of the v-operation it can be shown that D is a GCD domain if and only if $(a_1, a_2, \ldots, a_n)_v$ is principal for $a_i \in D^*$. For more on v-ideals and GCD domains the reader can consult Gilmer [12]. An integral domain is called a *v-coherent domain* if $aD \cap bD$ is a v-ideal of finite type for each $a, b \in D^*$, and a v-coherent domain in which $((a, b)(a, b)^{-1})^{-1} = D$ for each pair $a, b \in D^*$ is called a *Prüfer v-multiplication domain* ($PVMD$) [25].

Gauss' Lemma, initially proved for the ring of integers $\mathbb{Z}$, gives that the product of primitive polynomials is again primitive. Here a polynomial $f(X)$ is *primitive* if the coefficients of f have no nonunit common factor. Let us call an integral domain D a *Gauss' Lemma (GL) domain* if the product of every pair of primitive polynomials over D is again primitive. It can be shown that in a GL domain every irreducible element (atom) is a prime. So if every nonzero nonunit element of D is a product of atoms (e.g., D has the ascending chain condition on principal ideals) and D has the GL property, then D is a UFD. Now the GL property is also satisfied by GCD domains. Note that over a GCD domain a polynomial $f(X)$ is primitive if and only if $(A_f)_v = D$, here A_f denotes the fractional ideal generated by coefficients of f. Call a polynomial f *superprimitive* if $(A_f)_v = D$. Now a superprimitive polynomial is obviously primitive and as we indicate in the sequel the product of two superprimitive polynomials is again superprimitive. This led to the study of domains in which every primitive polynomial is superprimitive, such domains are called *PSP domains*. These considerations were made by Arnold and Sheldon [6]. Some further strengthenings of the Gauss' Lemma property were made in Anderson [5] of which the strongest is: For every nonzero integral ideal A the associated v-ideal A_v is the intersection of all the integral principal ideals containing A. He called this property the *IP property*. All these strengthenings of Gauss' Lemma were studied in [2] where the following diagram was given: GCD⇒ IP⇒ PSP⇒ GL⇒ atoms are prime (AP), with examples showing that no implications could be reversed. In Section 3 we show that while the IP property is independent of the Schreier property, the rest are implied by the Schreier property. Our example is somewhat simpler than that of [5, Example 3.2] and leads to a whole class of pre-Schreier domains that do not satisfy the IP property.

One application of Gauss' Lemma is to show that if D is a GCD domain then so is the polynomial ring $D[X]$. This is done by showing that the product of two primitive polynomials is again primitive. Since every polynomial $f \in D[X]^*$ is expressible as a product $f = af_1$ where $a \in D^*$ and f_1 is primitive, combining this with Gauss' Lemma gives that every pair of elements of D has a GCD. In [9] Cohn showed that if D is a Schreier domain then so is $D[X]$. Products of primitive polynomials are primitive over a Schreier domain but this does not help in the proof that $D[X]$ is a Schreier domain. In order to accomplish this task Cohn used a Nagata type theorem. In Section 4 we redo Cohn's proof from a slightly different point of view and show that using the notion of LCM splitting sets we can use a Nagata type theorem to directly show that if D is a GCD domain then so is $D[X]$.

2 Polynomial characterization of Schreier domains

Recently there has been some activity in using quadratic polynomials to determine the divisibility properties of integral domains, see for instance Waterhouse [22] and Rush [20]. This reminded us of a paper by Malik, Mott and Zafrullah [16]. In [16] the following basic result was proved: *Let D be a PVMD with quo-*

tient field K such that for each pair $a, b \in D^*$ there exist $c, d \in K^*$ such that $(a, b)^{-1} = (c, d)_v$. Then D is a GCD domain if and only if every irreducible quadratic polynomial over D is prime. The proof of the if part was based on the idea of taking a quadratic polynomial $g(X)$ of a specific type in $D[X]$ that splits in $K[X]$ as a product of two linear polynomials and concluding that $g(X)$ must be a product of two linear polynomials in $D[X]$ and then using the rather stringent conditions to show that $aD \cap bD$ is indeed principal for each pair $a, b \in D^*$. One aim of this section is to use the current state of knowledge to prove the following improved version of the above theorem.

Theorem 2.1. A v-coherent domain D is a GCD domain if and only if every quadratic polynomial in $D[X]$ that splits in $K[X]$ as a product of linear polynomials also splits in $D[X]$ as a product of linear polynomials.

The proof of this theorem consists of bringing together various pieces complete the picture. We shall use the notions of Schreier and pre-Schreier domains. Several characterizations of pre-Schreier domains are found in McAdam and Rush [18] and some more in [23] where the term pre-Schreier was coined. We use the following characterization of pre-Schreier domains.

Proposition 2.2. [23, Theorem 1.1(3)] An integral domain D is a pre-Schreier domain if and only if for each pair $a, b \in D^*$ and for all $x_1, x_2, \ldots, x_r \in aD \cap bD$ there exists $d \in aD \cap bD$ such that $d|x_i$ for all $i = 1, 2, \ldots, r$.

This proposition can also be traced back to [18] but as stated in [23] it tells us that if D is a pre-Schreier domain and if a, b are two nonzero elements in D then every finite subset $\{x_1, x_2, \ldots, x_r\}$ of $aD \cap bD$ has a common factor in $aD \cap bD$. This is all we need to prove the following proposition.

Proposition 2.3. An integral domain D is a GCD domain if and only if D is pre-Schreier and v-coherent.

Proof. Let a, b be any two nonzero elements of D. Since D is v-coherent there exist $x_1, x_2, \ldots, x_n \in D$ such that $aD \cap bD = (x_1, x_2, \ldots, x_n)_v$. Since D is pre-Schreier and since $x_1, x_2, \ldots, x_n \in aD \cap bD$ there is an $m \in aD \cap bD$ such that $m|x_i$ for each $i = 1, \ldots, n$. Thus $(x_1, x_2, \ldots, x_n) \subseteq mD$ and hence $aD \cap bD = (x_1, x_2, \ldots, x_n)_v \subseteq mD$. But since $m \in aD \cap bD$ we have $aD \cap bD = mD$. So $aD \cap bD$ is principal for each pair $a, b \in D^*$ and hence D is a GCD domain. For the only if part note that a GCD domain is Schreier [9] and a GCD domain is obviously v-coherent.□

To complete the proof of Theorem 2.1, we need to show that if every quadratic polynomial over D that splits in $K[X]$ as a product of linear polynomials also splits in $D[X]$ as a product of linear polynomials, then D is pre-Schreier. We want the pre-Schreier property, but we do not care if it is delivered by considering a specific type of quadratic polynomials or considering all quadratic polynomials over D. This is where special quadratics over D come in. These are polynomials of the type $f(X) = a(X + \frac{m}{a})(X + \frac{n}{a})$ such that $a, m, n \in D$ and $f(X) \in D[X]$ which means that $a|mn$ in D. These polynomials are sure to split in $K[X]$. These polynomials were recently used by Waterhouse [22]. Now Rush [20] has shown that an integral domain D is a pre-Schreier domain if and only if every special quadratic polynomial in $D[X]$ is expressible as a product of linear polynomials in $D[X]$. Since [20] has not yet appeared and since the procedure involved is pretty, we include the statement and proof below.

Theorem 2.4. An integral domain D is pre-Schreier if and only if every special quadratic $f(X) = a(X+\frac{m}{a})(X+\frac{n}{a})$ with $a, m, n \in D$ such that $f(X) \in D[X]$ is expressible as a product of linear polynomials $f(X) = (\alpha X+\beta)(\gamma X+\delta)$ where $\alpha, \beta, \gamma, \delta \in D$.

Proof. Let $a|mn$ in D, $a, m, n \in D^*$ and construct the following quadratic polynomial: $aX^2 + (m+n)X + \frac{mn}{a}$ in $D[X]$. By the condition there exist $\alpha, \beta, \gamma, \delta \in D$ such that $aX^2+(m+n)X+\frac{mn}{a} = (\alpha X+\beta)(\gamma X+\delta)$. Comparing coefficients we get (a) $a = \alpha\gamma$, (b) $m+n = \alpha\delta+\beta\gamma$, and (c) $\frac{mn}{a} = \beta\delta$. Then (d) set $\alpha\delta = u$, $\beta\gamma = v$.

(1) From (b) and (d) we get $u+v = m+n$ or $u+v-m = n$.

(2) From (a), (c) and (d) we get $mn = uv$.

Substituting for n in (2) we get $m(u+v-m) = uv$ which gives $m(u+v)-m^2 = uv$ or $m^2-(u+v)m+uv = 0$. Factoring we get $(m-u)(m-v) = 0$. If $m = u$ then $m = \alpha\delta$ which forces $n = \beta\gamma$ (cf (b) above). But $a = \alpha\gamma$ by (a). Thus in this case $a|mn$ implies that $a = \alpha\gamma$ where $\alpha|m$ and $\gamma|n$.

If on the other hand $m = v$, we get $m = \beta\gamma$, $n = \alpha\delta$, and $a = \alpha\gamma$. So, in both cases $a = rs$ where $r|m$ and $s|n$. Now as $a, m, n \in D^*$ are arbitrary, D is a pre-Schreier domain.

Conversely, suppose that D is pre-Schreier and let for $a, m, n \in D^*$, $f(X) = a(X+\frac{m}{a})(X+\frac{n}{a}) \in D[X]$. Then $a|mn$ and since D is pre-Schreier we have $a = rs$ such that $r|m$ and $s|n$. Let $m = rk$ and $n = sh$, $k, h \in D$. Then $f(X) = a(X+\frac{m}{a})(X+\frac{n}{a}) = rs(X+\frac{rk}{rs})(X+\frac{sh}{rs}) = rs(X+\frac{k}{s})(X+\frac{h}{r}) = (s(X+\frac{k}{s}))(r(X+\frac{h}{r})) = (sX+k)(rX+h)$ and both factors are in $D[X]$.□

Proof of Theorem 2.1. Let D be a v-coherent domain such that every quadratic polynomial over D that splits in $K[X]$ also splits in $D[X]$ as a product of two linear polynomials. Then in particular D is a v-coherent domain such that every special quadratic in $D[X]$ is a product of linear polynomials in $D[X]$. By Theorem 2.4, D is a pre-Schreier v-coherent domain and by Proposition 2.3, D is a GCD domain.

Conversely, let D be a GCD domain and let $f(X) = aX^2+bX+c$ be any quadratic polynomial in $D[X]$ such that in $K[X]$ we have $f(X) = aX^2+bX+c = (rX+t)(sX+u)$ where $r, s, t, u \in K$. Since D is a GCD domain $GCD(a, b, c)$ exists and is a unit in $K[X]$, so we can divide out by the GCD and assume that $\text{GCD}(a, b, c) = 1$, that is, we can assume that $f(X)$ is a primitive polynomial. But then it is well known that $D[X]$ is a GCD domain and every primitive polynomial in $D[X]$ is a product of primes in $D[X]$ [12, Theorem 34.10]. Now f can not be a prime in $D[X]$, for then it is a prime in $K[X]$. Hence f is a product of linear polynomials in $D[X]$.□

>From a slightly different angle we can state the following result.

Corollary 2.5. For a v-coherent domain D the following are equivalent.

(1). For every polynomial $f \in D[X]$ of positive degree with $f = gh$ where $g, h \in K[X]$ are both of positive degree, there exist $r, s \in D[X]$ such that r has the same degree as either g or h and $f = rs$.

(2). For every polynomial $f \in D[X]$ of positive degree, f has the same number of linear factors in $D[X]$ as it has in $K[X]$.

(3). Every polynomial of positive degree in $D[X]$ that has a linear factor in $K[X]$ has a linear factor in $D[X]$.

(4). Every quadratic polynomial of $D[X]$ that splits as a product of linear polynomials in $K[X]$ also splits in $D[X]$.

(5). Every special quadratic of Rush in $D[X]$ has a linear factor in $D[X]$.

(6). D is a GCD domain.

Proof. That (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) is obvious with or without the hypothesis that D is v-coherent. Now by the proof of Theorem 2.1, (5) $\Rightarrow$ (6) under the assumption that D is v-coherent. To show that (6) $\Rightarrow$ (1) note that $D[X]$ is a GCD domain [12, Theorem 34.10] and that a GCD domain is Schreier [9]. Let $f = gh$ where $g, h \in K[X]$ and let α, $\beta \in D^*$ be such that $\alpha g, \beta h \in D[X]$. So if $r = \alpha\beta$, then $rf = (\alpha g)(\beta h)$. This implies that $r|(\alpha g)(\beta h)$. But since $D[X]$ is a Schreier domain $r = st$ such that $s|\alpha g$ and $t|\beta h$ in $D[X]$. This forces $f = \frac{(\alpha g)}{s}\frac{(\beta h)}{t}$.□

It may be noted that the presence of v-coherence is not the only condition that makes the six conditions of Corollary 2.5 equivalent. As Waterhouse [22] pointed out, the presence of (5) makes every irreducible element of D a prime. Thus if we replace "D is v-coherent" by "D is atomic" the six conditions of Corollary 2.5 will still be equivalent. The well known examples of v-coherent domains are Prüfer domains, PVMD's, and coherent domains. Interested readers may state their own corollaries as "A Prüfer domain is a Bezout domain if and only if...", "A PVMD or coherent domain is GCD if and only if...", etc.

Now it often happens with a bunch of equivalent conditions with an accompanying hypothesis, some conditions are more equivalent than others. Here conditions (1)-(3) of Corollary 2.5 are all equivalent to a fourth one: D is Schreier. Let us put this as the following proposition.

Proposition 2.6. For an integral domain D with quotient field K the following are equivalent.

(1). For every polynomial $f \in D[X]$ of positive degree with $f = gh$ where $g, h \in K[X]$ are both of positive degree there exist $r, s \in D[X]$ such that r has the same degree as either g or h and $f = rs$.

(2). For every polynomial $f \in D[X]$ of positive degree f has the same number of linear factors in $D[X]$ as it has in $K[X]$.

(3). Every polynomial of positive degree in $D[X]$ that has a linear factor in $K[X]$ has a linear factor in $D[X]$.

(4). D is integrally closed and every quadratic polynomial of $D[X]$ that splits as a product of linear polynomials in $K[X]$ also splits in $D[X]$.

(5). D is integrally closed and every special quadratic in $D[X]$ has a linear factor in $D[X]$.

(6). D is a Schreier domain.

Proof. As before (1) $\Rightarrow$ (2) $\Rightarrow$ (3). For (3) $\Rightarrow$ (4) note that if $u \in K$ is an element integral over D then u satisfies a monic polynomial $f(X)$ of minimal degree over D. Now note that (3) guarantees a linear factor of $f(X)$ in $D[X]$, say $f(X) = (\alpha X - \beta)g(X)$. But since f is monic α must be a unit and so we can put $\alpha = 1$ to get $f(X) = (X - \beta)g(X)$ where $\beta \in D$ and $g(X) \in D[X]$. Now $0 = f(u) = (u - \beta)g(u)$ and $g(u) \neq 0$ because of the minimality condition on f. So $u = \beta \in D$. Having seen that D must be integrally closed, we again note that if $f \in D[X]$ is a quadratic that has a linear factor in $K[X]$ then (3) guarantees a linear factor of f in $D[X]$. But this forces f to be a product of linear polynomials in $D[X]$. This completes the proof of (3) $\Rightarrow$ (4). Now (4) $\Rightarrow$ (5) is clear and for (5) $\Rightarrow$ (6) we may use Theorem 2.4 and the fact that a Schreier domain is an integrally closed pre-Schreier domain. Finally for (6) $\Rightarrow$ (1) note [9] that if D is Schreier then so is $D[X]$ and use the same proof as that of (6) $\Rightarrow$ (1) in Corollary 2.5.$\square$

Note 2.7. The equivalences: (4) $\Leftrightarrow$ (5) $\Leftrightarrow$ (6) can be found in [20] and (1) $\Leftrightarrow$ (6) is in [18] while (2) and (3) appear to be new.

Now looking at (2) one may ask: What if the quotient field of the Schreier domain is algebraically closed? The answer is quite apparent in light of Proposition 2.6, yet for the record we state the following result.

Corollary 2.8. An integral domain D is a Schreier domain with algebraically closed quotient field K if and only if every polynomial $f(X)$ over D is a product of linear polynomials.

Proof. If every polynomial over D is expressible as a product of linear polynomials then the same holds over the quotient field K and so K is algebraically closed. Next if every polynomial over D splits as a product of linear polynomials then every monic over D has a linear factor. So D is integrally closed. Finally since every polynomial splits over D, so do all the special quadratics and by Proposition 2.6, D is a Schreier domain. Conversely if D is Schreier with algebraically closed quotient field, then Proposition 2.6 ensures that every polynomial over D splits as a product of linear polynomials.$\square$

It may be noted that any Schreier domain D with algebraically closed quotient field is a domain in which there are no irreducible elements, because for each nonzero nonunit $\alpha \in D^*$, $X^2 - \alpha$ is a product of two linear polynomials over D which gives $\alpha = (\sqrt{\alpha})^2$ a product of two nonunits. Such domains are called *antimatter domains* in [10].

The ring of algebraic integers is an example of a Schreier domain with algebraically closed quotient field. It would be interesting to find other examples. The rather obvious other examples are valuation domains with algebraically closed quotient fields. There are indeed plenty of them. For a start let P be a prime ideal in the ring of algebraic integers. Then R_P is a valuation domain with the property that every polynomial over D_P splits as a product of linear polynomials over D_P. This of course is not a groundbreaking discovery but does lead us to a slightly better result.

Proposition 2.9. Let D be an integral domain with quotient field $K \neq D$. Then there exists a valuation domain V containing D such that every polynomial over D is expressible as a product of linear polynomials over V.

Proof. Let $\overline{K}$ be the algebraic closure of K and let D^a be the integral closure of D in $\overline{K}$. Then D^a is integrally closed and so has a valuation overring V containing D^a and hence V is a valuation domain with quotient field $\overline{K}$. Now as $D \subseteq D^\alpha \subseteq V \subseteq \overline{K}$ we have the result.□

This proposition leads to the question: Must each D^α as constructed in Proposition 2.9 be a Schreier domain? This question seems to be a bit hard and so is left to an interested reader. Up to now we have seen examples of Bezout domains each with algebraically closed quotient field, is there an example of a non-Bezout Schreier domain with algebraically closed quotient field? The answer is yes as shown in the following example.

Example 2.10. (Coykendall) Let $\mathbb{C}$ denote the complex numbers, $D = \mathbb{C}[X]$, $K = \mathbb{C}(X)$, $\overline{K}$ the algebraic closure of K and let D^a be the integral closure of D in $\overline{K}$. Further let M be a maximal ideal in D^a and let $\overline{\mathbb{Q}}$ be the algebraic closure of the rationals $\mathbb{Q}$. Then $R = \overline{\mathbb{Q}} + MD^a_M$ is a Schreier domain with quotient field $\overline{K}$ and $S = \mathbb{Q} + MD^a_M$ is a pre-Schreier domain. Neither domain is a Bezout domain.

Illustration. Let M be a maximal ideal of D^a. Since D^a is a Prüfer domain [15, Theorem 101] D^a_M is a valuation domain. We show that $D^a_M = \mathbb{C} + MD^a_M$. For this note that $D^a_M/(MD^a_M) \approx D^a/M$ which is algebraic over $D/(M \cap D)$. But $M \cap D$ is a nonzero prime ideal of $D = \mathbb{C}[X]$ and so $M \cap D$ is generated by a linear polynomial, because $\mathbb{C}$ is algebraically closed. But then $D/(M \cap D) \approx \mathbb{C}$. This makes $D^a_M/(MD^a_M)$ algebraic over $\mathbb{C}$. Next $D^a_M \supseteq \mathbb{C}$ and so $D^a_M \supseteq \mathbb{C} + MD^a_M$. To see that the containment is not proper let $v \in D^a_M \setminus \mathbb{C} + MD^a_M$. Then for no $m \in MD^a_M$ can we have $v + m \in \mathbb{C}$. But since $D^a_M/(MD^a_M)$ is algebraic over $\mathbb{C}$, $v + MD^a_M$ should be algebraic over $\mathbb{C}$, which is algebraically closed and so $v + MD^a_M = c + MD^a_M$ for some $c \in \mathbb{C}$.

Now set $R = \overline{\mathbb{Q}} + MD^a_M$ and note that R is a $D + M$ construction of [7] and that R is integrally closed [7, Theorem 2.1 (b)]. Next, since $(MD^a_M)^{-1} = D^a_M$, R is a pseudo valuation domain. (Recall that a quasi-local domain (D, M) with maximal ideal M is called a *pseudo valuation domain (PVD)* if for all ideals A, B of D we have $A \subseteq B$ or $BM \subseteq AM$ [14].) So, MD^a_M is a divisorial ideal of R. Finally since D^a_M is a valuation domain with algebraically closed quotient field, D^a_M is antimatter and so for each nonzero $x \in MD^a_M$ we have $r, s \in MD^a_M$ such that $x = rs$. This forces $(MD^a_M)^2 = MD^a_M$. But a PVD with maximal ideal idempotent is a pre-Schreier ring by Proposition 3.7 of this article and an integrally closed pre-Schreier domain is Schreier. That R and S are not Bezout follows from [7, Theorem 2.1 (i)].

Now, a special quadratic in $D[X]$ looks very factorable. Is there an example that shows that sometimes a special quadratic may not be factorable in $D[X]$? In other words, is there an element that is not primal? The answer can be given in a number of ways. For instance in a non-integrally closed one-dimensional local domain no irreducible element is prime but an irreducible primal element is prime. But here we give a concrete example that has some other uses.

Example 2.11. Let (V, M) be a valuation domain with quotient field $K \neq V$ such that $M^{-1} = V$ and let L be a nontrivial extension of K. Then in $D = V + XL[[X]]$ the element X is not primal and so for $\alpha \in L\backslash K$ the special quadratic $X(t + \frac{\alpha X}{X})(t + \frac{X/\alpha}{X})$ cannot factor in the polynomial ring $D[t]$.

Illustration. Factoring the special quadratic requires that $X = rs$ where $r|\alpha X$ and $s|X/\alpha$. Exactly one of r, s can be of degree one in X. Say r is of degree one in X then $s(0) \neq 0$ and hence must be (an associate of) an element of V, so there is no harm in assuming that $s = s(0)$. On the other hand $r = \gamma X|\alpha X$. So $\frac{\alpha}{\gamma}$ must be an associate of an element of V. Since $\alpha \in L\backslash K$, γ must be in $L\backslash K$. But this is impossible because $rs = X = \gamma X s$ which forces $\gamma s = 1$ or $s = \frac{1}{\gamma} \in L\backslash K$. This is a contradiction delivering the conclusion that we cannot write $X = rs$ such that $r|\alpha X$ and $s|X/\alpha$.

The above example may also serve as an example of a domain that is not pre-Schreier.

3 Mixing Gauss' Lemma and the Schreier property

There are many ways unique factorization domains can be generalized. Of these, two ways in which an atomic domain is still a UFD are the following.

(1). Cohn's Approach

We know from [9] that a GCD domain is a Schreier domain, a Schreier domain is pre-Schreier, and in pre-Schreier domains every atom (irreducible element) is prime. So a pre-Schreier domain has the atoms are primes (AP) property. This gives the following diagram:

(A) GCD $\Rightarrow$ Schreier $\Rightarrow$ pre-Schreier $\Rightarrow$ AP property

As we shall find out in the sequel, none of these implications can be reversed.

(2). The Gauss' Lemma Approach

We note that a UFD D satisfies Gauss' lemma: Over a UFD the product of two primitive polynomials over D is again primitive.

By a primitive polynomial $f(X) \in D[X]$ here we mean the coefficients of $f(X) = \Sigma_{i=0}^{n} a_i X^i$ do not have nonunit common factor. Stated in ring-theoretic terms, a polynomial is primitive if the ideal generated by the coefficients of $f(X)$, that is, $A_f = (a_0, a_1, \ldots, a_n)$ is not contained in any proper principal ideal. Let us call a domain D a GL domain (GL for Gauss' Lemma) if over D the product of two primitive polynomials is again primitive. It is well known that domains that satisfy Gauss' Lemma do not have to be GCD domains. Here's an indirect way of finding out. A primitive polynomial $f \in D[X]$ is called superprimitive if $(A_f)^{-1} = D$, or equivalently $(A_f)_v = D$. Not all primitive polynomials are superprimitive. Here is a quick example.

Example 3.1. Take a one dimensional local domain D which is not a valuation domain. Then D must have two irreducible elements a, b which are not associates of each other. Then $h = aX + b$ is primitive but not superprimitive.

Illustration. Suppose that $(a, b)_v = D$. Now since D is one dimensional local, according to Kaplansky [15, Theorem 108], there is a least positive integer n such that $a|b^n$. Now note that $(a) = (a, b^n) = (a, b^n)_v = (a, ab^{n-1}, b^n)_v = (a, (ab^{n-1}, b^n)_v)_v = (a, b^{n-1}(a, b)_v)_v = (a, b^{n-1})_v$ which means that $a|b^{n-1}$ contradicting the minimality of n.

Example 3.1 gives us an example of a primitive polynomial that is not superprimitive. On the other hand a superprimitive polynomial is always primitive. For if not, let f be a superprimitive polynomial over D and suppose that there is $d \in D$ such that $A_f \subseteq dD$. Then, $D = (A_f)_v \subseteq dD$ and so d must be a unit. We say that $a, b \in D$ are *v-coprime* if $(a, b)_v = D$, i.e., $aX + b$ is superprimitive.

A domain in which a primitive polynomial is superprimitive is said to have the PSP property. Now all these concepts have been well studied in Arnold and Sheldon [6] and at a later date in Anderson and Quintero [2]. But for the sake of completeness we shall make necessary statements and give their proofs. First let us see how PSP works, but for this we need the Dedekind-Merten's Lemma: Let f and g be two polynomials in $D[X]$ and suppose that $deg(g) = m$. Then $(A_f)^{m+1}A_g = (A_f)^m A_{fg}$ [12].

Proposition 3.2. A PSP domain is a GL domain and a GL domain has the AP property.

Proof. Let f and g be two primitive polynomials over the PSP domain D. So $(A_f)_v = D = (A_g)_v$. We have to show that $(A_{fg})_v = D$. Now by Dedekind-Merten, we have $(A_f)^{m+1}A_g = (A_f)^m A_{fg}$. Hence $((A_f)^{m+1}A_g)_v = ((A_f)^m A_{fg})_v$. Now using the v-product rule we see that the LHS is $((A_f)^{m+1}A_g)_v = (((A_f)_v)^{m+1}(A_g)_v)_v = D$. Now the RHS using the v-product is $((A_f)^m A_{fg})_v = (((A_f)_v)^m A_{fg})_v = (A_{fg})_v$. Thus $(A_{fg})_v = D$.

Next suppose that D satisfies the GL property and assume that D has an atom a that is not a prime. Then there exist $b, c \in D$ such that $a|bc$ but a does not divide either of b, c. Then $bX + a$ and $cX + a$ are both primitive. But in the product $(bX + a)(cX + a) = bcX^2 + (b + c)aX + a^2$ every coefficient is divisible by a, a contradiction forcing the conclusion that there is no atom in a GL domain D that is not a prime.□

Now looking at the domain $D = V + XL[[X]]$ constructed in Example 2.11 it is easy to see that a polynomial $f \in D[X]$ is primitive if and only if f has a unit coefficient. So D has PSP and hence GL, yet D is not pre-Schreier and hence not GCD.

$>$From the above considerations of the Gauss' Lemma approach we can construct the following picture:

(B) GCD property $\Rightarrow$ PSP property $\Rightarrow$ GL property $\Rightarrow$ AP property

Now we have two diagrams, (A) and (B); so the question arises: Both of them end at the same property (AP), are they equivalent?

We now use some more of our knowledge of the pre-Schreier property and Example 2.11 to show that the pictures (A) and (B) are two different pictures. First let us recall from [24] that if $A = (a_0, a_1, a_2, \ldots, a_n)$ is a finitely generated nonzero ideal in a Schreier domain such that $a_0, a_1, a_2, \ldots, a_n$ have no nonunit common factor then $A^{-1} = D$. In other words, if $f(X) = \Sigma_{i=0}^{n} a_i X^i$ is a primitive polynomial over a Schreier domain then $(A_f)^{-1} = D$. That is, a Schreier domain has the PSP property. Now because in [24] the author only used the characterization of pre-Schreier domains the same proof applies to the following proposition.

Proposition 3.3. A pre-Schreier domain has PSP. In particular, in a pre-Schreier domain every pair of coprime elements is v-coprime.

Now dropping the word "property" to save space we have the following picture

$$\begin{array}{ccccccc} \text{GCD} & \Rightarrow & \text{Schreier} & \Rightarrow & \text{pre-Schreier} & \Rightarrow & \text{AP} \\ & & \Downarrow & \swarrow\!\!\!\swarrow & \Downarrow & & \\ \text{GCD} & \Rightarrow & \text{PSP} & \Rightarrow & \text{GL} & \Rightarrow & \text{AP} \end{array}$$

or GCD$\Rightarrow$ Schreier $\Rightarrow$ pre-Schreier $\Rightarrow$ PSP $\Rightarrow$ GL $\Rightarrow$ AP

Now we prepare to show that this picture is the best possible. Recall that an integral domain D has the IP property if for each nonzero integral ideal A of D, A_v is the intersection of principal integral ideals . That is, D has IP if for each integral nonzero ideal A we have $A_v = \cap dD$ where dD ranges over principal integral ideals of D that contain A. As indicated in [2], the IP property implies the PSP property. So if we give an example of a domain D that is not a pre-Schreier domain but has the AP property we have established that the PSP property does not imply the pre-Schreier property. This can be amply established if we show that the ring constructed in Example 2.11 satisfies the IP property. For this we first note that the construction in Example 2.11 is a $D + M$ construction from a valuation domain of the type $L + M$ where L is a field. Now we quote the following result from Anderson [5].

Theorem 3.4. [5, Proposition 2.3 (1)] Let V be a nontrivial valuation domain of the form $K + M$ where K is a field and M is the maximal ideal of V. Let R be the subring $D + M$ where D is a proper subring of the field K. Then R satisfies the IP property if and only if D satisfies the IP property and D is not a field.

Corollary 3.5. The construction in Example 2.11 is an IP domain that is not Schreier.

This means that IP $\not\Rightarrow$ pre-Schreier. Now our study of mixing Gauss' Lemma and the (pre-) Schreier property would be complete if we can show either that IP$\not\Rightarrow$ pre-Schreier or that pre-Schreier$\Rightarrow$ IP.

To show that IP$\not\Rightarrow$ pre-Schreier we need a quasi-local pre-Schreier domain that has a divisorial maximal ideal that is not principal. (Any maximal ideal that is contained in a principal integral ideal would have to be that principal ideal!). Let us look for quasi-local domains that have divisorial maximal ideals which are not principal. It is easy to see that if (D, M) is a PVD that is not a valuation domain then $M = D \cap xD$ for some $x \in K$ [23, Proposition 4.1]. Now since an intersection of principal fractional ideals is a v-ideal, in light of the above discussion we have the following proposition.

Proposition 3.6. A PVD (D, M) that is not a valuation domain does not satisfy the IP property.

Now all we need is a pre-Schreier (or Schreier) PVD that is not a valuation domain. Here we quote parts of Theorem 4.4 of [23].

Proposition 3.7. [23, Theorem 4.4 (i) and (iii)] Let (D, M) be a PVD that is not a valuation domain. Then D is a pre-Schreier domain if and only if $M^2 = M$.

Now it so happens that the rings R, S of Example 2.10 have the precise credentials for the Schreier property and for the pre-Schreier property and both are PVD's. Now we include, just for record, the following well known result.

Proposition 3.8. A PVD (D, M) with the GCD property is a valuation domain.

Proof. We show that if D is a PVD that is not a valuation domain then it cannot be a GCD domain. If D is a PVD that is not a valuation domain then (a) the maximal ideal M is divisorial as we have remarked earlier. So for all $A \subseteq M$ we have $A_v \subseteq M$ and (b) there must be a pair of elements a, b in D such that $a \nmid b$ and $b \nmid a$. Now suppose on the contrary that D has the GCD property. Then $d = GCD(a, b)$ implies that $a = a_1 d, b = b_1 d$ where a_1 and b_1 are coprime. But since a GCD domain is Schreier, a_1, b_1 coprime implies a_1, b_1 v-coprime, i.e., $(a_1, b_1)_v = D$. This means that at least one of a_1, b_1 is a unit. But then $a|b$ or $b|a$, a contradiction.□

$>$From Example 2.10 we have a pre-Schreier PVD that is not a valuation domain and hence not a GCD domain by Proposition 3.8.

Theorem 3.9. The pre-Schreier (Schreier) property implies neither GCD nor IP property.

We must point out that the case of IP$\not\Rightarrow$ Schreier has been dealt with in [5], but as our procedure provides a whole class of such domains and is simpler, we have included it here.

Now let us recall the last diagram and adjust it with our current state of knowledge. We had:

GCD ⇒ Schreier ⇒ pre-Schreier ⇒ PSP ⇒ GL ⇒ AP

>From [5] we recall that GCD ⇒ IP and from the above discussion we know that IP ⇏ pre-Schreier and pre-Schreier ⇏ IP. This gives us the following picture:

$$\begin{array}{ccccccccccc} \text{GCD} & \Rightarrow & \text{Schreier} \Rightarrow \text{pre-Schreier} & \Rightarrow & \text{PSP} & \Rightarrow & \text{GL} & \Rightarrow & \text{AP} \\ & \searrow & & \nearrow & & & & & \\ & & \text{IP} & & & & & & \end{array}$$

Looking at the diagram we note that IP is the strongest property in the Gauss' Lemma mode of generalizing GCD domains and IP has no relationship with the pre-Schreier property as Examples 2.10 and Example 2.11 show. Finally with reference to [2] and [5] no arrows can be reversed.

Let us get back again to Example 2.11. In [8] a somewhat elaborate construction is employed to construct an antimatter domain that is not pre-Schreier. Below we offer Example 2.11 as an easier alternative.

Proposition 3.10. The construction $V + XL[[X]]$ in Example 2.11 is an antimatter domain.

Proof. Recall from [10] that a valuation domain (V, M) is an antimatter domain if and only if $M^{-1} = V$ and this was required in Example 2.11. So in $V + XL[[X]]$ all the nonunits that are associates of elements of V do not have any atomic factor. This leaves us with associates of elements of the form lX^n where $l \in L^*$. Now clearly lX^n is not an atom for $n > 1$. But lX is not an atom because for every nonunit $v \in V^*$ we have $lX = v(\frac{lX}{v})$.□

4 Schreier property or Gauss' Lemma?

Having seen how the Schreier property fares in comparison with various stronger forms of Gauss' Lemma one may ask: Is Gauss' Lemma a must?

The answer seems to be: Gauss' Lemma is a part of our heritage and so is a must. However, there are situations in which Gauss' Lemma does not help and there are results that help us bypass Gauss' Lemma. Nagata's theorem for UFD's shows that if D is a UFD then so is $D[X]$ without even touching a polynomial and Cohn's "Nagata's theorem for Schreier domains" does precisely the same for Schreier domains. Before we go on further it seems pertinent to recall a few facts along with Cohn's Nagata type theorem.

Recall that a nonzero element x is called *completely primal* if every factor of x is again primal. By definition every factor of a completely primal element is again completely primal. According to Cohn [9] the product of two completely primal elements is again completely primal [9, Lemma 2.5]. Thus, with the same proof as in [9, Lemma 2.5], we have the following lemma.

Lemma 4.1. If S is a multiplicative set of a domain D generated by a set of completely primal elements and if S' is the saturation of S then S' consists of completely primal elements which are either in S or factors of elements of S.

Theorem 4.2. [9, Theorem 2.6] Let R be an integrally closed integral domain and S a multiplicative subset of R.

(i) If R is a Schreier domain, then so is R_S.

(ii) (Nagata type theorem) If R_S is a Schreier domain and S is the saturation of the set generated by a set of completely primal elements of R, then R is a Schreier domain.

Remarks 4.3. (1). Note that our wording of (ii) of Theorem 4.2 is slightly different from that of Cohn's actual theorem. This is because Cohn's statement is slightly vague. His proof works quite nicely if we assume that Cohn identifies S with its saturation (cf Lemma 4.1 here).

(2). In the proof of [9, Theorem 2.6], Cohn does not use the assumption that R is integrally closed. So, Theorem 4.2 above also holds for pre-Schreier domains. For the sake of clarity we restate this theorem below.

Theorem 4.4. (Cohn's Theorem for pre-Schreier domains). Let D be an integral domain and S a multiplicative set of D.

(i) If D is pre-Schreier, then so is D_S.

(ii) (Nagata type theorem) If D_S is a pre-Schreier domain and S is the saturation of the set generated by a set of completely primal elements of D, then D is a pre-Schreier domain.

Now the main use for Gauss' Lemma is to show that if D is a GCD domain and if X is an indeterminate over D then $D[X]$ is a GCD domain. The proof that $D[X]$ is a GCD domain then involves noting that every nonconstant polynomial $f(X) \in D[X]$ can be written as $f(X) = af_1(X)$, where $f_1(X)$ is a primitive polynomial etc. In the case of Schreier domains we cannot write every nonconstant polynomial $f(X) \in D[X]$ as $af_1(X)$ where $f_1(X)$ is primitive. For it can be shown that this is equivalent to D being a GCD domain. Theorem 4.2 (Theorem 2.6 of [9]) holds an answer to this but for that we need to prepare a little. This preparation is essentially to promote our view of the pre-Schreier property; we believe that completely primal elements are the building blocks of the pre-Schreier property.

Lemma 4.5.[3, Theorem 3.1] A nonzero element c of D is completely primal if and only if for all $d \in D^*$, $cD \cap dD$ is locally cyclic. Moreover, if $\{c_1, c_2, ..., c_n\}$ is a set of nonzero completely primal elements of D then $\cap c_i D$ is locally cyclic.

Proof. This lemma is part (3) of Theorem 3.1 of [3], except for the moreover part. For that we proceed by induction. Note that the case of $n = 1$ is clear and $n = 2$ follows from the first part. Suppose that $\cap c_i D$ is locally cyclic for $i = n - 1$. Let $x_1, x_2, \ldots, x_r \in \cap_{i=1}^{n} c_i D$. Then $x_1, x_2, \ldots, x_r \in \cap_{i=1}^{n-1} c_i D$ and by the induction hypothesis there is $t \in \cap_{i=1}^{n-1} c_i D$ such that $x_1, x_2, \ldots, x_r \in tD$. But then $x_1, x_2, \ldots, x_r \in tD \cap c_n D$ and since c_n is completely primal there is $s \in tD \cap c_n D$ such that $x_1, x_2, \ldots, x_r \in sD$. But $sD \subseteq tD \cap c_n D \subseteq \cap_{i=1}^{n} c_i D$.□

Lemma 4.6. A nonzero element c of D is completely primal if and only if $c|a_i b_j$ for $i = 1, \ldots, m$, and $j = 1, \ldots, n$ implies that $c = c_1 c_2$ where $c_1|a_i$ for all i and $c_2|b_j$ for all j.

Proof. Suppose that c is completely primal and that $c|a_ib_j$ for $i = 1, \ldots, m$, and $j = 1, \ldots, n$. Now since c is primal in D and $a_i, b_j \in D$ we have for $i = 1$ the following picture: $c|a_1b_1, a_1b_2, a_1b_3, \ldots, a_1b_n \Rightarrow c = c_1^{(j)}c_2^{(j)}$ where $c_1^{(j)}|a_1$ and $c_2^{(j)}|b_j$ for $j = 1, \ldots, n$. Since each of $c_1^{(j)}$ is completely primal because c is and since $a_1, c \in \cap c_1^{(j)}D$, we conclude by Lemma 4.5 that there is an $r \in D$ such that $c_1^{(j)}|r|a_1, c$. Thus $c = r(\frac{c}{r}) = c_1^{(j)}c_2^{(j)}$. Since for each j we have $c_1^{(j)}|r$ we conclude that $(\frac{c}{r})|c_2^{(j)}|b_j$ for each $j = 1, \ldots, n$. Thus we have shown that $c|a_1b_1, a_1b_2, a_1b_3, \ldots, a_1b_n$ implies that $c = r_1s_1$ where $r_1|a_1$ and $s_1|b_j$ for all j. Using the same procedure we can show for each $i = 1, \ldots, m$ that $c|a_ib_1, a_ib_2, a_ib_3, \ldots, a_ib_n$ implies that $c = r_is_i$ where $r_i|a_i$ and $s_i|b_j$ for all $j = 1, \ldots, n$.

Since $c, b_1, b_2, ..., b_n \in \cap s_iD$ and since the s_i are completely primal (being factors of c) we conclude that there is a t such that $s_i|t$ and $t|c, b_j$. Set $c = t(\frac{c}{t}) = r_is_i$ and note that $s_i|t$ for all i which forces $(\frac{c}{t})|r_i$ for all $i = 1, \ldots, m$. But $r_i|a_i$ for $i = 1, \ldots, m$. Thus we conclude that $c|a_ib_j$ for $i = 1, \ldots, m$ and $j = 1, \ldots, n$ implies that $c = c_1c_2$ where $c_1|a_i$ for all i and $c_2|b_j$ for all j.

Conversely, suppose that $c|a_ib_j$ for $i = 1, \ldots, m$, and $j = 1, \ldots, n$ implies that $c = c_1c_2$ where $c_1|a_i$ for all i and $c_2|b_j$ for all j and consider $(c) \cap (x)$ for any $x \in D^*$. We show that $(c) \cap (x)$ is locally cyclic. For this let $x_1, x_2, \ldots, x_r \in (c) \cap (x)$. We can write $x_i = xh_i$ and conclude that $c|xh_i$. Then by our assumption $c = c_1c_2$ such that $c_1|x$ and $c_2|h_i$ for $i = 1, \ldots, r$. But then $m = xc_2 \in (c) \cap (x)$ and $m|x_i$ for each i. Thus $(c) \cap (x)$ is locally cyclic for each $x \in D^*$ and by Lemma 4.5 c is completely primal.□

Lemma 4.7. If D is integrally closed then every completely primal element of D is a completely primal element of the polynomial ring $D[X]$.

Proof. Let c be a completely primal element of D and suppose that for $f, g \in D[X]$ we have $c|fg$. Let $f = \Sigma_{i=0}^{m} a_iX^i$ and let $g = \Sigma_{j=0}^{n} b_jX^j$. Recall that D is integrally closed if and only if for$h, k \in K[X]$, $A_{hk} \subseteq D$ implies that $A_hA_k \subseteq D$ (see [17, Theorem 1.5] for an easy proof). Replacing f by $\frac{f}{c}$ in the above statement we conclude that in our case $c|fg$ implies that $c|a_ib_j$ for $i = 1, \ldots, m$ and $j = 1, \ldots, n$. Now since c is completely primal in D and $a_i, b_j \in D$ we have by Lemma 4.6 that $c = c_1c_2$ where $c_1|a_i$ and $c_2|b_j$. But then $c|fg$ in $D[X]$ implies that $c = c_1c_2$ such that $c_1|f$ and $c_2|g$. This shows that every completely primal element of an integrally closed D is primal in $D[X]$. But then every factor of a completely primal element in D is completely primal in D and hence is primal in $D[X]$. □

Theorem 4.8.[9, Theorem 2.7] Let D be an integral domain. Then D is a Schreier domain if and only if $D[X]$ is a Schreier domain.

Proof. Let $D[X]$ be a Schreier domain. Then since $D[X]$ is integrally closed we have D integrally closed. Now let $c, a, b \in D^*$ such that $c|ab$ in D. But then $c|ab$ in $D[X]$ and so $c = c_1(X)c_2(X)$ where $c_1(X)|a$ and $c_2(X)|b$. But by degree considerations $c_i(X) \in D$. So every nonzero element of D is primal in D. Conversely let D be Schreier. Then by Lemma 4.7, all the elements of D^*

are completely primal in $D[X]$ and indeed D^* is a saturated multiplicative set in $D[X]$. Now $D[X]_{D^*} = K[X]$ is a PID and hence a Schreier domain. Thus Theorem 4.2, forces $D[X]$ to be a Schreier domain.□

Now the role of Gauss' Lemma in showing that if D is a GCD domain then so is $D[X]$ is the following. It shows that if D is a GCD domain then the product of any two primitive polynomials over D is again primitive. But as we have already noted in Section 3, if D is (pre-) Schreier then D has PSP, so we can conclude that if D is Schreier then the products of primitive polynomials in $D[X]$ is again primitive. Now recall from [9] that a GCD domain is a Schreier domain. So by Theorem 4.8, if D is a GCD domain then $D[X]$ is at least a Schreier domain. It is easy to see that over any domain D the factors in $D[X]$ of a primitive polynomial are again primitive. Now using the GCD property of D we can show that every polynomial $f(X) = \Sigma_{i=0}^{m} a_i X^i$ in $D[X]^*$ can be written as $f(X) = d(\Sigma_{i=0}^{m} \frac{a_i}{d} X^i)$ where $d = GCD(a_1, a_2, ..., a_m)$ and $(\Sigma_{i=0}^{m} \frac{ai}{d} X^i)$ is primitive. Note that a primitive polynomial of degree 0 is a unit in D. Next because of the degree restrictions every primitive polynomial can be expressed as a finite product of irreducible elements, and in a Schreier domain every irreducible element is a prime. So every primitive polynomial in $D[X]$ is a product of primes and the task of showing that $D[X]$ is a GCD domain can be accomplished as in the last part of [12, Theorem 34.10]. Recent work has provided a more efficient method of showing that if D is a GCD domain then so is $D[X]$. But for that we need to prepare a little. A saturated multiplicative set S of D is called a splitting multiplicative set of D if every nonzero element d of D can be written as $d = st$ where $s \in S$ and t is v-coprime to every member of S. A splitting multiplicative set S is called an lcm splitting set if in addition, for every pair of elements $a, b \in S$ we have $aD \cap bD$ in S. For a detailed study of these concepts the reader may consult [1] where the following result is attributed (on page 30) to [13] and to [19]. ([19] came from Schexnayder's dissertation [21] which was cited in [13].)

Theorem 4.9. Let S be an lcm splitting set of an integral domain D. If D_S is a GCD domain, then so is D.

Based on these observations we state and prove the following well known theorem.

Theorem 4.10. If D is a GCD domain and X an indeterminate over D, then $D[X]$ is a GCD domain.

Proof. Note that by Theorem 4.8, $D[X]$ is a Schreier domain, and that in a Schreier domain every pair of coprime elements is v-coprime (cf Proposition 3.3). Now we show that D^* is a splitting set in $D[X]$. Let $f \in D[X]^*$. If $\deg(f) = 0$, then $f = f \cdot 1$, so let us assume that $\deg(f) > 0$. Then because D is a GCD domain, $f = ag$ where $a \in D^*$ is the GCD of the coefficients of f and g is a primitive polynomial and hence is coprime to every element of D^*. But since $D[X]$ is Schreier, g is v-coprime to every element of D^*. This establishes that D^* is a splitting set of $D[X]$. That D^* is an lcm splitting set follows from the fact that for all $x, y \in D^*$ we have $xD \cap yD$ principal and that leads to $xD[X] \cap yD[X]$ principal. Now to complete the proof we note that $(D[X])_{D^*} = K[X]$ a PID and hence a GCD domain, forcing by Theorem 4.9, $D[X]$ to be a GCD domain.□

There are of course other results that give the same conclusion as in Theorem 4.10. Look up for example [4, Theorem 2.2] where it is shown that $D[X]$ is a GCD domain if and only if D^* is a splitting set of $D[X]$.

References

[1] D.D. Anderson, D.F. Anderson, and M. Zafrullah, Splitting the t-class group, J. Pure Appl. Algebra **74** (1991), 17-37.

[2] D.D. Anderson and R. Quintero, Some generalizations of GCD-domains, *Factorization in Integral Domains*, 439–480, Lecture Notes in Pure and Appl. Math., 189, Dekker, New York, 1997.

[3] D.D. Anderson and M. Zafrullah, P.M. Cohn's completely primal elements, *Zero-dimensional Commutative Rings*, 115–123, Lecture Notes in Pure and Appl. Math., 171, Dekker, New York, 1995.

[4] D.D. Anderson and M. Zafrullah, Splitting sets in integral domains, Proc. Amer. Math. Soc. **129** (2001), 2209-2217.

[5] D.F. Anderson, Integral v-ideals, Glasgow Math. J. **22** (1981), 167–172.

[6] J. Arnold and P. Sheldon, Integral domains that satisfy Gauss's lemma, Michigan Math. J. **22** (1975), 39–51.

[7] E. Bastida and R. Gilmer, Overrings and divisorial ideals of rings of the form D+M, Michigan Math. J. **20** (1973), 79–95.

[8] G. Brookfield and D. Rush, An antimatter domain that is not pre-Schreier, preprint.

[9] P.M. Cohn, Bezout rings and their subrings, Proc. Cambridge Phil. Soc. **64** (1968), 251-264.

[10] J. Coykendall, D. Dobbs, and B. Mullins, On integral domains with no atoms, Comm. Algebra **27** (1999), 5813–5831.

[11] D. Dobbs, Coherence, ascent of going-down, and pseudo-valuation domains, Houston J. Math. **4** (1978), 551–567.

[12] R. Gilmer, *Multiplicative Ideal Theory*, Dekker, New York, 1972.

[13] R. Gilmer and T. Parker, Divisibility properties in semigroup rings, Michigan Math. J. **21** (1974), 65-86.

[14] J. Hedstrom and E. Houston, Pseudo-valuation domains, Pacific J. Math. **75** (1978), 137–147.

[15] I. Kaplansky, *Commutative Rings*, Allyn and Bacon, Boston, 1970.

[16] S. Malik, J. Mott, and M. Zafrullah, The GCD property and irreducible quadratic polynomials, Internat. J. Math. Sci. **9** (1986), 749–752.

[17] J. Mott, B. Nashier, and M. Zafrullah, Contents of polynomials and invertibility, Comm. Algebra **18** (1990), 1569-1583.

[18] S. McAdam and D. Rush, Schreier rings, Bull. London Math. Soc. **10** (1978), 77–80.

[19] J. Mott and M. Schexnayder, Exact sequences of semi-value groups, J. Reine Angew Math. **283/284** (1976), 388-401.

[20] D. Rush, Quadratic polynomials, factorization in integral domains and Schreier domains in pullbacks, Mathematika, to appear.

[21] M. Schexnayder, *Groups of Divisibility*, Dissertation, Florida State University, 1973.

[22] W. Waterhouse, Quadratic polynomials and unique factorization, Amer. Math. Monthly **109** (2002), 70–72.

[23] M. Zafrullah, On a property of pre-Schreier domains, Comm. Algebra **15** (1987), 1895-1920.

[24] M. Zafrullah, Well-behaved prime t-ideals, J. Pure Appl. Algebra **65** (1990), 199–207.

[25] M. Zafrullah, Putting t-invertibility to use, *Non-Noetherian Commutative Ring Theory*, 429–457, Math. Appl., 520, Kluwer Acad. Publ., Dordrecht, 2000.

Department of Mathematics
The University of Iowa
Iowa City, IA 52242
dan-anderson@uiowa.edu

57 Colgate Street
Pocatello, ID 83201
zafrullah@lohar.com