Domains over which polynomials split

Muhammad Zafrullah

Abstract. An integrally closed integral domain R is a Schreier ring if its group of divisibility is a Riesz group, i.e., for all $a, b, c \in R\backslash\{0\}$ $a \mid bc$ implies that $a = xy$ where $x \mid b$ and $y \mid c$. We show that R is a Schreier ring with algebraically closed field of fractions if and only if every polynomial of degree greater than one over R splits, i.e., is a product of linear polynomials.

1. Introduction

Surfing the internet I came across an online discussion that led me in the end to a pretty survey article [MM]. This paper, which is to appear in the American Mathematical Monthly, starts off with the question (Question 1.1): Can every polynomial with integer coefficients be factored into (not necessarily monic) linear terms each with algebraic integer coefficients? Aside from the historical perspective provided in the above mentioned article the answer to the question can be given in two ways each with very little amount of Mathematics: (1) Yes, because the ring of algebraic integers is a GCD domain with algebraically closed quotient field. (2) Yes, because the ring of algebraic integers is a Schreier ring with algebraically closed quotient field. It so happens, as we find out in the course of the second answer, that R is a Schreier domain with algebraically closed quotient field if and only if every polynomial of positive degree over R is either linear or a product of linear polynomials over R. For want of a better name let us call these integral domains "Domains over which polynomials split" or "Algebraically closed domains". We use algebraically closed domains to construct a new set of examples of integral domains that have no irreducible elements. These integral domains were studied in [CDM] as "antimatter" domains. A more detailed study of various new constructions of antimatter domains is under way.

2. Simpler Approach

An integral domain R is a GCD domain if for all $a, b \in R\backslash\{0\}$ GCD(a, b) exists and is in R.

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Theorem 1. *Let R be a GCD domain with quotient field K. Then K is algebraically closed if and only if in $R[X]$ every polynomial $f(X) = \sum r_i X^i$ of degree greater than 0 is a linear polynomial or a product of linear polynomials.*

Proof. Let R be a GCD domain with algebraically closed quotient field K. Let $f(X) = \sum_{i=0}^{n} r_i X^i$ be a polynomial of degree greater than 0 in $R[X]$.Recall the following information from Theorem 34.10 of Gilmer [G]:

(i) $R[X]$ is a GCD domain, (ii) $f(X) = r \sum_{i=0}^{n} s_i X^i = r f_1(X)$ where $r = GCD(r_1, ...r_n), s_i = \frac{r_i}{r}$ and the expression $f(X) = r f_1(X)$ is unique up to associates where $r \in R$ and $f_1(X)$ is a primitive polynomial, (iii) $f_1(X)$ is a product of prime polynomials of $R[X]$ (iv) Each irreducible polynomial of $R[X] \backslash R$ is a prime in $K[X]$. So to prove that $f(X) = \sum_{i=0}^{n} r_i X^i$ is a product of linear polynomials it is enough to show that every irreducible factor of $f_1(X)$ is a linear polynomial. For this let $h(X)$ be an irreducible factor of $f_1(X)$. Then $h(X)$ is a prime polynomial such that $h(X)$ is a prime polynomial in $K[X]$. But since K is algebraically closed the only prime polynomials in $K[X]$ are linear. But this forces $h(X)$ to be linear.

The converse is straightforward because for any integral domain R if every polynomial of positive degree is either linear or a product of linear polynomials then the same is true of all polynomials of positive degree over K.

Corollary 1. *Let D be an integral domain that is contained in a GCD domain R such that the quotient field of R is algebraically closed and let X be an indeterminate over R then every polynomial in X of degree greater than 0 over D is a linear polynomial or can be expressed as a product of linear polynomials with coefficients in R.*

Proof. Note that $D[X] \subseteq R[X]$.

Corollary 2. *(Answer (1)) Let $f(X)$ be a polynomial of positive degree over a subring of the ring of algebraic integers then $f(X)$ is a linear polynomial or is a product of linear polynomials over the ring of algebraic integers.*

Proof. The ring A of algebraic integers is a Bezout domain (every finitely generated ideal is principal) and hence is a GCD domain. Moreover the quotient field of A is the algebraic closure of Q the field of rational numbers.

3. More Advanced Approach

An integrally closed integral domain R is called a Schreier domain if its group of divisibility is a Riesz group i.e. for every triple of nonzero elements a, b, c of R, $a \mid bc$ implies that $a = xy$ where $x \mid b$ and $y \mid c$. This notion was introduced by Cohn [C], and from the definition it is easy to see that in a Schreier domain $a \mid b_1 b_2 ... b_n$ implies $a = x_1 x_2 ... x_n$ where $x_i \mid b_i$ and that a GCD domain is a Schreier domain. It was also shown in [C] that if R is Schreier then so is $R[X]$.

Theorem 2. *Let R be a Schreier domain then the quotient field K of R is algebraically closed if and only if every polynomial of positive degree over R is a linear polynomial or a product of linear polynomials with coefficients in R.*

Proof. Let $f(X)$ be a nonlinear polynomial over R. Then $f(X) = f_1 f_2 .... f_n$ over $K[X]$ where each of f_i is linear, because K is algebraically closed. Now for each i there is $a_i \in R \backslash \{0\}$ such that $a_i f_i \in R[X]$. Let $a = a_1 a_2 ... a_n$. Then

$af(X) = \prod a_i f_i(X)$. But since in the Schreier domain $R[X]$, $a \mid \prod a_i f_i(X)$ we have $a = \prod b_i$ where $b_i \mid a_i f_i(X)$. But then $f(X) = \prod \frac{a_i f_i(X)}{b_i}$ a product of linear polynomials over $R[X]$.

Conversely if R is any domain such that every $f(X) \in R[X] \backslash R$ is a product of linear polynomials over R then the same would happen over the quotient field K.

Corollary 3. *Let D be an integral domain that is contained in a Schreier domain R such that the quotient field of R is algebraically closed and let X be an indeterminate over R then every polynomial in X of degree greater than 0 over D is a linear polynomial or can be expressed as a product of linear polynomials with coefficients in R.*

Then the inevitable Corollary.

Corollary 4. *(Answer (2)). Let $f(X)$ be a polynomial of positive degree over a subring of the ring of algebraic integers then $f(X)$ is a linear polynomial or is a product of linear polynomials over the ring of algebraic integers.*

Proof. A GCD domain is Schreier.

Now the question is: What are the domains over which every polynomial of positive degree is either linear or a product of linear factors? The answer is "good old Schreier domains" with quotient field algebraically closed.

Theorem 3. *Let R be an integral domain such that every polynomial of positive degree over R is a linear polynomial or a product of linear polynomials. Then R is a Schreier domain with quotient field algebraically closed.*

Proof. That R is integrally closed follows from the fact that every monic polynomial of positive degree over R is a product of linear (and of course) monic linear polynomials over R. For the remainder we use a beautiful argument due to David E. Rush [R], note that in particular every quadratic polynomial in $R[X]$ is expressible as a product of two linear polynomials over $R[X]$. Now let $a \mid bc$ in R, $a, b, c \in R \backslash \{0\}$ and construct the following quadratic polynomial: $aX^2 + (b+c)X + \frac{bc}{a}$ in $R[X]$. By the condition there exist $\alpha, \beta, \gamma, \delta \in R$ such that $aX^2 + (b+c)X + \frac{bc}{a} = (\alpha X + \beta)(\gamma X + \delta)$. Comparing coefficients we get $a = \alpha\gamma$...(a), $b + c = \alpha\delta + \beta\gamma$...(b), and $\frac{bc}{a} = \beta\delta$...(c), and set $\alpha\delta = u$, $\beta\gamma = v$...(d).

From (b) and (d) we get $u + v = b + c$ or $u + v - b = c$ ....(1)

From (a), (c) and (d) we get $bc = uv$ ....(2)

Substituting for c in (2) we get $b(u + v - b) = uv$ which gives $b(u + v) - b^2 = uv$ or $b^2 - (u + v)b + uv = 0$. Factorizing we get $(b - u)(b - v) = 0$. If $b = u$ then $b = \alpha\delta$ which forces $c = \beta\gamma$ (cf (b) above). But $a = \alpha\gamma$ by (a). Thus in this case $a \mid bc$ implies that $a = \alpha\gamma$ where $\alpha \mid b$ and $\gamma \mid c$.

If on the other hand $b = v$ we get $b = \beta\gamma$ and $c = \alpha\delta$ and $a = \alpha\gamma$. So, in both cases $a = xy$ where $x \mid b$ and $y \mid c$. Now as $a, b, c \in R \backslash \{0\}$ are arbitrary, R is a Schreier domain.

As stated in [CDM], there is a reason to look for integral domains that do not have any irreducible elements. If K is a field, Q^+ denotes non-negative rational numbers, and if $S = \{X^q : q \in Q^+\}$ then the monoid domain $K[S]$ being an ascending union of $R_n = K[X^{\frac{1}{n!}}]$ of PID's is a Bezout domain. If K is algebraically closed it was mentioned in [Za], albeit indirectly, that $K[S]$ contains no irreducible elements. The reason being that if $f \in K[S] \backslash K$ is irreducible in R_n then it is a polynomial of higher degree in some R_m, $m > n$ and so must be a expressible as a product of more linear polynomials in R_m. Using the same idea we state the following corollary.

Corollary 5. *Let R be a Schreier domain without any irreducible elements whose quotient field K is algebraically closed and let $S = \{X^\alpha : \alpha \in Q^+\}$, then the semigroup ring $R[S]$ is a Schreier ring that has no irreducible elements.*

Proof. Let $R_n = R[X^{\frac{1}{n!}}]$. Then each of R_n is a Schreier ring being a polynomial ring over a Schreier ring and according to [C] an ascending union of Schreier rings is again Schreier. In each of R_n the nonzero nonunits are products of linear polynomials by Theorem 2. So if there is an irreducible element $f \in R = \cup R_n$ and if k is the least such that f is irreducible in R_k but then for $m > k$, f is a polynomial of higher degree in R_m and hence has more linear factors in R_m, but each linear factor in R_m is a nonunit in R_m and hence in R.

Here is an in direct result.

Corollary 6. *Every Schreier domain with algebraically closed field of fractions is an antimatter domain.*

Remark 1. *The procedure of David Rush mentioned in the proof of Theorem 3 was used in [R] in the characterization of integral domains R whose group of divisibility is a Riesz group. These domains were dubbed as pre-Schreier domains in [Z]. So an integrally closed pre-Schreier domain is a Schreier domain. In [Z] it was shown that in some respects pre-Schreier domains can be characterized in almost the same way as the Schreier domains and among other things it includes for demonstration purposes an example of a non-integrally closed pre-Schreier domain S and it was demonstrated that the polynomial ring $S[X]$ was not pre-Schreier. (I found out in July 2003 that [R] was to appear in Mathematika, but for some reason Mathematika is not appearing.)*

Remark 2. *Professor Rush suggests combining Theorems 2 and 3 into: Theorem. Let R be an integral domain with quotient field K. Then each $f \in R[X]$ is a product of linear polynomials in $R[X]$ if and only if R is Schreier and K is algebraically closed.*

I trust the readers can do that in their minds, I do not want to disturb the flow of the article.

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57 Colgate Street, Pocatello, ID 83201
E-mail address: mzafrullah@usa.net
URL: http://www.lohar.com