t-SPLITTING SETS IN INTEGRAL DOMAINS

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Abstract. Let D be an integral domain with quotient field K. A multiplicative subset S of D is a t-splitting set if for each $0 \neq d \in D$, $dD = (AB)_t$ for some integral ideals A and B of D, where $A_t \cap sD = sA_t$ for all $s \in S$ and $B_t \cap S \neq \emptyset$. A t-splitting set S of D is a t-lcm (resp., Krull) t-splitting set if $sD \cap dD$ is t-invertible (resp., sD is a t-product of height-one prime ideals of D) for all nonunits $s \in S$ and $0 \neq d \in D$. Let S be a t-splitting set of D, $\mathcal{T} = \{A_1 \cdots A_n | A_i = d_i D_S \cap D$ for some $0 \neq d_i \in D\}$, and $D_{\mathcal{T}} = \{x \in K | xC \subseteq D$ for some $C \in \mathcal{T}\}$. We show that S is a t-lcm (resp., Krull) t-splitting set if and only if $D_{\mathcal{T}}$ is a PVMD (resp., Krull domain), if and only if every finite type integral v-ideal (resp., every integral ideal) of D intersecting S is t-invertible. We also show that $D \setminus \{0\}$ is a t-splitting set in $D[X]$ if and only if D is a UMT-domain and that every nonempty multiplicative subset of $D[X]$ contained in $G = \{f \in D[X] | (A_f)_v = D\}$ is a t-lcm t-complemented t-splitting set of $D[X]$. Using this, we give several Nagata-like theorems.

1. INTRODUCTION

Let D be an integral domain. A saturated multiplicative subset S of D is called a *splitting set* if for each $0 \neq d \in D$, $d = sa$ for some $s \in S$ and $a \in D$ with $aD \cap s'D = as'D$ for all $s' \in S$. If S is a splitting set, then the set $T = \{x \in D | (x, s)_v = D$ for all $s \in S\}$ is also a splitting set called the *m-complement of* S. A splitting set S is said to be an *lcm splitting set* if $sD \cap dD$ is principal for all $s \in S$ and $0 \neq d \in D$. Following [6], we call a (not necessarily saturated) multiplicative subset S of D a *t-splitting set* if for each $0 \neq d \in D$, $dD = (AB)_t$ for some integral ideals A and B of D, where $A_t \cap sD = sA_t$ for all $s \in S$ and $B_t \cap S \neq \emptyset$ and 't' is the well-known t-operation, equivalently, for each $0 \neq d \in D$, $dD_S \cap D$ is t-invertible [6, Proposition 3.1]. Let $\overline{S}$ be the saturation of a multiplicative subset S of D. Then $D_S = D_{\overline{S}}$ [21, Proposition 5.1], and hence S is a t-splitting set if and only if $\overline{S}$ is a t-splitting set. One aim of this paper is to demonstrate that by adopting

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this t-splitting approach, we can provide direct and simple proofs of results that otherwise need elaborate constructions or longer proofs. Briefly, in this article we study various aspects of t-splitting sets and find for example that $D\backslash\{0\}$ is a t-splitting set of $D[X]$ if and only if D is a so-called UMT-domain. Parallel to the notion of an lcm-splitting set, we introduce the notion of a *t-lcm t-splitting set* and use it to prove a number of more general Nagata-like theorems in the spirit of [3, Proposition 4.3]. For example, we show that the set $G = \{f \in D[X] \mid (A_f)_v = D\}$ is a t-lcm t-splitting set in $D[X]$ and give a simple proof of the fact that D is a PVMD if $D[X]_G$ is a PVMD. To give a complete introduction, however, we need to introduce the reader to the notions involved.

Throughout this paper, D will denote an integral domain, K is the quotient field of D, $U(D)$ is the group of units in D, and $X^1(D)$ is the set of height-one prime ideals of D. Recall that for a nonzero fractional ideal I of D, $I^{-1} = \{x \in K | xI \subseteq D\}$, $I_v = (I^{-1})^{-1}$, and $I_t = \cup\{(a_1, \ldots, a_n)_v | 0 \neq (a_1, \ldots, a_n) \subseteq I\}$. We say that I is a *divisorial ideal* or *v-ideal* (resp., *t-ideal*) if $I_v = I$ (resp., $I_t = I$) and that I is a *finite type v-ideal* if $I = (a_1, \ldots, a_n)_v$ for some $0 \neq (a_1, \ldots, a_n) \subseteq I$. It is well known that every proper integral t-ideal is contained in some (necessarily prime) t-ideal maximal among proper integral t-ideals and that every prime ideal minimal over a t-ideal is a t-ideal, in particular, height-one prime ideals are t-ideals. The set of all maximal t-ideals of D is denoted by t-Max(D), while we say that D has *t-dimension one*, written t-dim$D = 1$, if each maximal t-ideal of D has height-one, i.e., t-Max$(D) = X^1(D)$. A fractional ideal I of D is said to be *t-invertible* if $(II^{-1})_t = D$. If a fractional ideal I is t-invertible, then I_t is a finite type v-ideal. The set of t-invertible fractional t-ideals of D forms an abelian group under the t-product $I * J = (IJ)_t$. The (t-)*class group* of D is $Cl(D)$ $-$ the abelian group of t-invertible fractional t-ideals of D, modulo its subgroup of principal fractional ideals. If D is a Krull domain, then $Cl(D)$ is the usual divisor class group of D; and if D is a Prüfer domain or one-dimensional integral domain, then $Cl(D) = Pic(D)$, the ideal class group (or Picard group) of D.

Let D be an integral domain. Then D is called a *weakly Krull domain* if $D = \cap_{P \in X^1(D)} D_P$ and the intersection has finite character. A nonzero element a of D is said to be *primary* if aD is a primary ideal of D. As in [10], we call D a *weakly factorial domain* (WFD) if each nonzero nonunit of D is a product of primary elements. In [10, Theorem], it was proved that D is a WFD if and only if D is a weakly Krull domain and $Cl(D) = 0$. An integral domain D is said to be a *generalized weakly factorial domain* (GWFD) if each nonzero prime ideal of D contains a primary element [13]. It is known that a WFD is a GWFD and a GWFD is a weakly Krull domain. Recall that an integral domain D is a UMT-domain if every upper to zero in $D[X]$ is a maximal t-ideal. Also, recall that an integral domain D is a *Prüfer v-multiplication domain* (PVMD) if each finite type v-ideal of D is t-invertible. It is well known that D is a PVMD if and only if D_P is a valuation domain for each maximal t-ideal P of D ([23, Theorem 5] or [26, Theorem 3.2]), if and only if D is an integrally closed UMT-domain [24, Proposition 3.2].

Let S be a t-splitting set of an integral domain D and let $\mathcal{T} = \{A_1 \cdots A_n | A_i = d_i D_S \cap D \text{ for some } 0 \neq d_i \in D\}$. Then $D_S = \cap\{D_P | P \in t\text{-Max}(D) \text{ and } P \cap S = \emptyset\}$, $D_{\mathcal{T}} = \cap\{D_P | P \in t\text{-Max}(D) \text{ and } P \cap S \neq \emptyset\}$, and $D = D_S \cap D_{\mathcal{T}}$ [6, Lemma 4.2 and Theorem 4.3]. A t-splitting set S of D is called the *t-complemented t-splitting set* if $D_{\mathcal{T}} = D_T$ for some multiplicative subset T of D, and the saturation of T is called the *t-complement of S*. It is clear that a splitting set is a t-complemented t-splitting set.

It is known that every (saturated) multiplicative subset of D is a t-splitting set if and only if D is a weakly Krull domain [6, page 8]; and that every saturated multiplicative subset of D is a splitting set if and only if D is a weakly Krull domain and $Cl(D) = 0$, if and only if D is a WFD [10, Theorem]. In Section 2, we show that every (saturated) multiplicative subset of D is a t-complemented t-splitting set if and only if D is a GWFD. Let S be a t-splitting set of D and $\mathcal{T} = \{A_1 \cdots A_n | A_i = d_i D_S \cap D \text{ for some } 0 \neq d_i \in D\}$. We also prove that $D_{\mathcal{T}} = \{x \in K | xC \subseteq D \text{ for some } C \in \mathcal{T}\}$ is a Krull domain if and only if each nonunit $s \in S$ is a t-product of (height-one) prime ideals of D; that if S is generated by principal primes, then S is a splitting set; and that $D \setminus \{0\}$ is a t-splitting set in $D[X]$ if and only if D is a UMT-domain.

As a t-splitting set analog of an lcm splitting set, we call a t-splitting set S of an integral domain D a *t-lcm t-splitting set* if for all $s \in S$ and $0 \neq d \in D$, $sD \cap dD$ is t-invertible. In Section 3, we prove that S is a t-lcm t-splitting set if and only if A is t-invertible for all finite type integral v-ideals A of D such that $A \cap S \neq \emptyset$, if and only if $D_{\mathcal{T}} = \{x \in K | xC \subseteq D \text{ for some } C \in \mathcal{T}\}$ is a PVMD; this is an analog of a similar result for lcm splitting sets [3, Proposition 2.4]. Let $G = \{f \in D[X] | (A_f)_v = D\}$ and $\emptyset \neq S \subseteq G$ be a multiplicative subset of $D[X]$. We also show that S is a t-lcm t-complemented t-splitting set of $D[X]$. As indicated above, we use this fact to provide the following Nagata-like theorem: D is a PVMD if and only if $D[X]_G$ is a PVMD. We also include a brief history of this result.

2. t-splitting sets

A Krull domain D is called an *almost factorial domain* if $Cl(D)$ is torsion (equivalently, if for each pair $0 \neq a, b \in D$, there exists an integer $n = n(a,b) \geq 1$ such that $a^nD \cap b^nD$ is principal [19, Proposition 6.8]). Almost factorial (Krull) domains were first studied by U. Storch [35].

Recall that every saturated multiplicative subset of an integral domain D is a splitting set (resp., t-splitting set) if and only if D is a WFD [10, Theorem] (resp., weakly Krull domain [6, page 8]) and that if D is a Krull domain, then D is an almost factorial domain if and only if every multiplicative subset of D is a t-complemented t-splitting set [6, page 15]. It was shown in [13, Proposition 3.1] that a Krull domain D is an almost factorial domain if and only if D is a GWFD, whence every multiplicative subset of a Krull domain D is a t-complemented t-splitting set if and only if D is a GWFD.

Theorem 2.1. *Let D be an integral domain. Then every (saturated) multiplicative subset of D is a t-complemented t-splitting set if and only if D is a GWFD.*

Proof. ($\Rightarrow$) Assume that every (saturated) multiplicative subset of D is a t-complemented t-splitting set. Then D is a weakly Krull domain [6, page 8], and hence t-dim$D = 1$ [9, Lemma 2.1]. So by [13, Theorem 2.1], it suffices to show that each $P \in X^1(D)$ is the radical of a principal ideal. To do this, let $P \in X^1(D)$ and $T = D \setminus P$. Then T is a t-complemented t-splitting set by assumption. Let S be the t-complement of T. Then S is a t-complemented t-splitting set and T is the t-complement of S (cf. [6, Theorem 4.3]). Let $F = \{Q \in X^1(D) | Q \cap S = \emptyset\}$ and $G = \{Q \in X^1(D) | Q \cap S \neq \emptyset\}$. Then $\cap_G D_Q = D_T = D_P$ [6, Lemma 4.2], and hence $G = \{P\}$. Thus, if $a \in P \cap S$, then $P = \sqrt{aD}$.

($\Leftarrow$) Assume that D is a GWFD. Let S be a multiplicative subset of D and $T = \{a_1 \cdots a_n |$ each $a_i \in D$ is primary and $\sqrt{a_iD} \cap S = \emptyset\}$. Recall that a GWFD is a weakly Krull domain [13, Corollary 2.3]; hence S is a t-splitting set [6, page 8], $D = \cap_{Q \in X^1(D)} D_Q$, and the intersection has finite character. In particular, $D_T = (\cap_{Q \in X^1(D)} D_Q)_T = \cap_{Q \in X^1(D)} (D_Q)_T$ [21, Proposition 43.5]. Also, note that for each $Q \in X^1(D)$, $Q \cap T \neq \emptyset \Leftrightarrow (D_Q)_T = K$, the quotient field of D, and $Q \cap T = \emptyset \Leftrightarrow Q \cap S \neq \emptyset$ by the construction of T (also refer to [13, Theorem 2.2]).

Thus $D_T = \cap_{Q \in X^1(D)} (D_Q)_T = \cap\{D_Q | Q \in X^1(D)$ and $Q \cap T = \emptyset\} = \cap\{D_Q | Q \in X^1(D)$ and $Q \cap S \neq \emptyset\} = \cap\{D_Q | Q \in t\text{-Max}(D)$ and $Q \cap S \neq \emptyset\}$, and hence S is a t-complement t-splitting set. □

Let D be an integral domain. An element $a \in D$ is called a *t-invertibility element* if for each integral ideal A of D, $a \in A$ implies that A is t-invertible. According to [22, Theorem 1.3], an element $a \in D$ is a t-invertibility element if and only if aD is a t-product of maximal t-ideals of D.

Let S be a multiplicative subset of an integral domain D. Recall that a prime ideal Q of D with $Q \cap S \neq \emptyset$ is said to *intersect S in detail* if $P \cap S \neq \emptyset$ for each nonzero prime ideal $P \subseteq Q$ of D. Let T be the m-complement of a splitting set S of D. In [3, Theorem 2.6], the authors proved that S is generated by principal primes if and only if D_T is a factorial domain. We now give a t-splitting set analog.

Theorem 2.2. *Let D be an integral domain with quotient field K, S a t-splitting set of D, $\mathcal{T} = \{A_1 \cdots A_n | A_i = d_i D_S \cap D$ for some $0 \neq d_i \in D\}$, and $G = \{P \in t$-Max$(D) | P \cap S \neq \emptyset\}$. Then the following statements are equivalent.*

1. *For each nonunit $s \in S$, sD is a t-product of prime ideals.*
2. *For each nonunit $s \in S$, sD is a t-product of height-one prime ideals.*
3. *$D_{\mathcal{T}} = \{x \in K | xC \subseteq D$ for some $C \in \mathcal{T}\}$ is a Krull domain.*
4. *Every integral ideal of D intersecting S is t-invertible.*

In this case, $X^1(D_{\mathcal{T}}) = \{PD_P \cap D_{\mathcal{T}} | P \in G\}$ and D_P is a DVR for each $P \in G$.

Proof. (1) ⇒ (2) It suffices to show that if P is a prime t-ideal of D with $P \cap S \neq \emptyset$, then P is of height-one. Let P be a prime t-ideal of D such that $P \cap S \neq \emptyset$ and let $s \in P \cap S$. Then $sD = (P_1 \cdots P_n)_t$ for some prime t-ideals P_i of D. Since sD is t-invertible, each P_i is t-invertible, and hence a maximal t-ideal of D [24, Proposition 1.3]. Also, since $sD \subseteq P$, $P_i \subseteq P$ for some P_i; so $P_i = P$. Thus P is a maximal t-ideal of D. Moreover, since P intersects S in detail [6, Lemma 4.2], P is of height-one.

(2) ⇒ (3) This appears in the proof of [6, Theorem 4.14].

(3) ⇒ (4) and (1) Assume that $D_{\mathcal{T}}$ is a Krull domain. If $P \in G$, then $(PD_{\mathcal{T}})_t \subsetneq D_{\mathcal{T}}$ since $P = P_t = (PD_S)_t \cap (PD_{\mathcal{T}})_t$ [6, Theorem 4.10] and $(PD_S)_t = D_S$. Let Q be a prime ideal of $D_{\mathcal{T}}$ minimal over $(PD_{\mathcal{T}})_t$ such that $Q \subseteq PD_P \cap D_{\mathcal{T}}$. Then Q is a prime t-ideal of $D_{\mathcal{T}}$ and $P = Q \cap D$. Thus Q has height-one, and hence

$(D_{\mathcal{T}})_Q = D_P = (D_{\mathcal{T}})_{PD_P \cap D_{\mathcal{T}}}$ is a DVR. In particular, since $D_{\mathcal{T}} = \cap_{P \in G} D_P$ [6, Theorem 4.3(2)], $X^1(D_{\mathcal{T}}) = \{PD_P \cap D_{\mathcal{T}} | P \in G\}$ [21, Corollary 43.9].

Let $s \in S$ be a nonunit of D. Then $D \cap sD_{\mathcal{T}} = D_S \cap sD_{\mathcal{T}} = (sD_S)_t \cap (sD_{\mathcal{T}})_t = sD$ [6, Theorem 4.10]. Since $D_{\mathcal{T}}$ is a Krull domain, the intersection $\cap_{P \in G} D_P$ has finite character; hence $sD_{\mathcal{T}} = \cap_{P \in G} sD_P = sD_{P_1} \cap \cdots \cap sD_{P_n} \cap D_{\mathcal{T}}$ for some $P_1, \ldots, P_n \in G$ (note that $n \geq 1$ since $sD_{\mathcal{T}} \subsetneq D_{\mathcal{T}}$). Also, since each D_{P_i} is a DVR, $sD_{P_i} = P_i^{e_i} D_{P_i} = (P_i^{e_i})_t D_{P_i}$ for some integer $e_i \geq 1$. Note that since each P_i is a maximal t-ideal of D, $(P_i^{e_i})_t$ is P_i-primary [4, Lemma 1]; so $(P_i^{e_i})_t D_{P_i} \cap D = (P_i^{e_i})_t$. Thus $sD = sD_{\mathcal{T}} \cap D = (P_1^{e_1})_t D_{P_1} \cap \cdots \cap (P_n^{e_n})_t D_{P_n} \cap D = (P_1^{e_1})_t \cap \cdots \cap (P_n^{e_n})_t = (P_1^{e_1} \cdots P_n^{e_n})_t$, where the last equality follows from the fact that each P_i is a maximal t-ideal. Moreover, since sD is t-invertible, each P_i is also t-invertible. This also shows that s is a t-invertibility element of D [22, Theorem 1.3], and thus every integral ideal of D intersecting S is t-invertible.

(4) $\Rightarrow$ (1) Let $s \in S$ be a nonunit of D. Then s is a t-invertibility element since every ideal of D containing s intersects S. Thus sD is a t-product of maximal t-ideals [22, Theorem 1.3]. □

Let us call the t-splitting set of Theorem 2.2 a *Krull t-splitting set*. Clearly an integral domain D is a Krull domain if and only if $D \setminus \{0\}$ is a Krull t-splitting set, and hence D is a Krull domain if and only if every (saturated) multiplicative subset of D is a Krull t-splitting set (cf. [6, page 8]).

Corollary 2.3. *An integral domain D is an almost factorial domain if and only if every* (*saturated*) *multiplicative subset of D is a t-complemented Krull t-splitting set.*

Proof. Recall that D is a Krull domain if and only if every nonzero integral ideal of D is t-invertible [27, Theorem 3.6] and that $D \setminus \{0\}$ is a t-splitting set; hence $D \setminus \{0\}$ is a Krull t-splitting set if and only if D is a Krull domain. Also, since a Krull domain is almost factorial if and only if it is a GWFD [13, Proposition 3.1], the result follows directly from Theorem 2.1 (or see [6, page 15]). □

Proposition 2.4. *Let D be an integral domain. Then the following statements are equivalent.*

(1) *D is a factorial domain.*

(2) *Every saturated multiplicative subset of D is generated by principal primes.*

(3) *Every saturated multiplicative subset of D has the property that for each integral ideal A intersecting S, A_t is principal.*

(4) *Every saturated multiplicative subset of D is a splitting set which is also a Krull t-splitting set.*

Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) Clearly $D \setminus \{0\}$ is a splitting set of D, and thus the results follow from the well-known fact that D is a factorial domain if and only if A_t is principal for all nonzero integral ideals A of D. (1) $\Leftrightarrow$ (4) As noted in the proof of Corollary 2.3, $D \setminus \{0\}$ is a Krull t-splitting set if and only if D is a Krull domain. Hence the result is an immediate consequence of [10, Theorem]. □

It is clear that a splitting set is a t-splitting set. In general, a t-splitting set need not be a splitting set. For example, let D be a weakly Krull domain with $Cl(D) \neq 0$. Then there is a saturated multiplicative subset S of D that is not a splitting set [10, Theorem]. But S is a t-splitting set since every multiplicative subset of a weakly Krull domain is a t-splitting set [6, page 8].

Corollary 2.5. *Let S be a t-splitting set of an integral domain D. If S is generated by principal primes, then S is a splitting set of D.*

Proof. Let the notation be as in Theorem 2.2 and let $\mathcal{P}$ be the set of principal primes of D generating S. Then $G = \{pD | p \in \mathcal{P}\}$ since a principal prime ideal is a maximal t-ideal [24, Proposition 1.3]. Thus for all $p \in \mathcal{P}$, D_{pD} is a DVR by Theorem 2.2, and hence $\cap_{n \geq 0} p^n D = 0$. Also, note that $D_{\mathcal{T}}$ is a Krull domain and $X^1(D_{\mathcal{T}}) = \{pD_{pD} \cap D_{\mathcal{T}} | p \in \mathcal{P}\}$; hence $\cap_\alpha p_\alpha D = 0$ for any infinite sequence $\{p_\alpha\}$ of nonassociated members of $\mathcal{P}$. Thus S is a splitting set [3, Proposition 2.6]. □

Recall that an integral domain D is called a *locally factorial domain* if $D_f = D[\frac{1}{f}]$ is factorial for all nonzero nonunit f of D. It is clear that a factorial domain is locally factorial. [16, Examples 3.1 and 3.2] show that a locally factorial domain need not be factorial. Another simple example is $\mathbb{Z}_{(2)} + X\mathbb{Q}[[X]]$ which has two proper overrings $\mathbb{Q}[[X]]$ and $\mathbb{Q}((X))$. The following corollary shows that a locally factorial domain which is not factorial does not contain a height-one principal prime.

Corollary 2.6. *Let D be a locally factorial domain that contains a height-one principal prime. Then D is a factorial domain.*

Proof. Let p be a height-one principal prime of D. If $D \setminus pD = U(D)$, then D is a local PID with maximal ideal pD; so we may assume that $D \setminus pD \neq U(D)$. Let

$f \in D \setminus pD$ be a nonunit of D. Then $(f,p)_v = D$, and hence $D = D_f \cap D_p$ [1, Lemma 2.1]. Since D is locally factorial, both D_f and D_p are factorial (and hence Krull domains). Hence D is a Krull domain [19, Proposition 1.4], and thus D is a factorial domain [1, Corollary 2.9]. (Another proof : Let $S = \{up^n \in U(D) |$ and $n \geq 0\}$. Then S is a splitting set by Corollary 2.5 since a multiplicative subset of a (weakly) Krull domain is a t-splitting set [6, page 8]. Thus $D_p = D_S$ being facotrial implies that D is factorial [3, Theorem 4.4].) □

Remark 2.7. Let D be a locally factorial Krull domain. Then a nonzero principal prime of D has height-one. Thus if D contains a nonzero principal prime, then D is a factorial domain by Corollary 2.6. This recovers [1, Corollary 2.9].

Theorem 2.8. *Let D be an integral domain and S a multiplicative subset of D such that D_S is a PID. Then S is a t-splitting set if and only if every prime ideal of D disjoint from S is t-invertible.*

Proof. ($\Rightarrow$) Assume that S is a t-splitting set of D. Let P be a prime ideal of D disjoint from S and $p \in P$ with $PD_S = pD_S$. Then $P = PD_S \cap D = pD_S \cap D$ is t-invertible [6, Corollary 2.3]. ($\Leftarrow$) Let G be the set of nonzero prime t-ideals of D disjoint from S. Let $0 \neq g \in D \setminus S$. Clearly $I_0 = gD$ is a t-invertible t-ideal and $I_0 \subseteq P_1$ for some $P_1 \in G$. Then $I_0 = (I_1P_1)_t$, where $I_1 = (I_0P_1^{-1})_t$ is again a t-invertible t-ideal. We repeat the procedure with I_1. After a finite number of steps, we get $I_0 = (P_1 \cdots P_n I_n)_t$ with $P_i \in G$ and I_n t-ideal with $I_n \cap S \neq \emptyset$. Indeed, since $I_0 \subseteq (P_1 \cdots P_n)_t D_S = P_1 \cdots P_n D_S$, we see that $n \leq$ the length of g in the PID D_S. Since each P_i is t-coprime with every element of S, so is $P_1 \cdots P_n$. □

Let D be an integral domain with quotient field K and X an indeterminate over D. Then for $f \in K[X]$, the *content* A_f *of* f denotes the fractional ideal of D generated by the coefficients of f. If A is a fractional ideal of $D[X]$, then the fractional ideal $c(A) = \sum_{f \in A} A_f$ of D is also called the *content* of A.

In [12, Theorem 2.2], it was proved that $S \subseteq D$ is a splitting set in $D[X]$ if and only if S is an lcm splitting set of D. Recall from [3, Proposition 2.4] that if T is the m-complement of a splitting set S, then S is an lcm splitting set if and only if D_T is a GCD-domain. Thus $D \setminus \{0\}$ is a splitting set in D[X] if and only if D is a GCD-domain [3, Example 4.7]. The following corollary is a t-splitting set analog of this result. This also gives another characterization of a UMT-domain.

Corollary 2.9. *Let D be an integral domain. Then $D \setminus \{0\}$ is a t-splitting set in $D[X]$ if and only if D is a UMT-domain.*

Proof. Clearly $D[X]_{D\setminus\{0\}}$ is a PID. Thus the result follows directly from Theorem 2.8 and the fact that D is a UMT-domain if and only if every upper to zero in $D[X]$ is t-invertible [24, Theorem 1.4]. □

We end this section with some applications of Corollary 2.9. Recall that a weakly Krull domain D is called a *generalized Krull domain* if D_P is a valuation domain for all $P \in X^1(D)$.

Corollary 2.10. *Let D be an integral domain.*

1. (cf. [7, Proposition 4.11]) *If $D[X]$ is a weakly Krull domain, then D is a weakly Krull UMT-domain.*
2. (cf. [7, Corollary 4.13]) *If D is an integrally closed weakly Krull domain, then $D[X]$ is a weakly Krull domain if and only if D is a generalized Krull domain.*

Proof. (1) This follows from the facts that $D[X]$ is a weakly Krull domain if and only if every multiplicative subset of $D[X]$ is a t-splitting set [6, page 8] and that $D\setminus\{0\}$ is a (saturated) multiplicative subset of $D[X]$. But then D is a UMT-domain by Corollary 2.9. That D is a weakly Krull domain can now be easily established.

(2) This follows from (1) above and the fact that an integrally closed UMT-domain is a PVMD [24, Proposition 3.2]. □

Now here is a serious question. It has been shown that $D \setminus \{0\}$ is a splitting set in $D[X]$ if and only if $D \setminus \{0\}$ is an lcm splitting set in $D[X]$ (if and only if D is a GCD-domain). How do we explain that result using the fact that D is a UMT-domain if and only if $D \setminus \{0\}$ is a t-splitting set in $D[X]$. The following corollary answers this question.

Corollary 2.11. *Let D be an integral domain.*

1. *If D is a UMT-domain, then D is a PVMD if and only if for every $0 \neq f \in D[X]$, $fD[X] = (A_f[X]B)_t$, where B is the t-product of uppers to zero.*
2. *$D \setminus \{0\}$ is a splitting set of $D[X]$ if and only if D is a GCD-domain.*

Proof. (1) Assume that for every $0 \neq f \in D[X]$, $fD[X] = (A_f[X]B)_t$, where B is the t-product of uppers to zero in $D[X]$. Then for each $0 \neq f \in D[X]$, A_f is t-invertible, and so every finitely generated ideal A of D is t-invertible, which makes D a PVMD. Conversely, let D be a PVMD and let $0 \neq f \in D[X]$; then A_f is t-invertible. Next, by the UMT condition we have that $fD[X] = (AB)_t$, where A and B are integral ideals of $D[X]$ such that $A \cap D \neq (0)$ and B is the t-product of uppers to zero in $D[X]$. Next, $fD[X] = (A_t B)_t \subseteq A_t$. Because a PVMD is integrally closed, we have $A_t = (c(A)[X])_t = (c(A))_t[X]$ [8, Theorem 3.2]. Now as $f \in fD[X] \subseteq A_t = (c(A))_t[X]$, we have $A_f \subseteq (c(A))_t$, and so $(A_f)_t \subseteq (c(A))_t$. For the reverse containment, note that $fD[X] \subseteq A_f[X]$ and so $(c(A)[X]B)_t \subseteq A_f[X]$. Since B is a t-product of maximal t-ideals that do not contain $A_f[X]$, we conclude that $(c(A)[X])_t \subseteq (A_f[X])_t$. This shows that $fD[X] = (A_f[X]B)_t = ((A_f)_t[X]B)_t$.

(2) Note that $D \setminus \{0\}$ being a splitting set requires that for each $0 \neq f \in D[X]$, $fD[X] = agD[X]$ where $a \in D$ and $g \in D[X]$ with $(g, s)_v = D[X]$ for all $0 \neq s \in D$. This means that $(A_g)_v = D$ and so $(A_f)_v = (A_{ag})_v = a(A_g)_v = aD$. The converse is easy to see. □

3. t-lcm t-splitting sets

Let D be an integral domain. A splitting set S of D is called a *t-lcm splitting set* if $sD \cap dD$ is t-invertible for all $s \in S$ and $0 \neq d \in D$. This concept was introduced by the third author at the conference held in Incheon, Korea (May, 2001). He also observed : *Let S be a splitting set of D and T the m-complement of S in D. Then (i) S is a t-lcm splitting set if and only if A is t-invertible for all finitely generated integral ideals A of D with $A \cap S \neq \emptyset$. (ii) If S is a t-lcm splitting set, then D_T is a PVMD. (iii) If S is a t-lcm splitting set, then D is a PVMD if and only if D_S is a PVMD, if and only if A is t-invertible for all finitely generated integral ideals A of D with $A \cap T \neq \emptyset$.* The purpose of this section is to study these concepts in a more general setting. To do this, we introduce a new concept "t-lcm t-splitting set", which is a generalization of a t-lcm splitting set.

Let S be a t-splitting set of an integral domain D. Then we will call S a *t-lcm t-splitting set* if $sD \cap dD$ is t-invertible for all $s \in S$ and $0 \neq d \in D$. Recall from [6, Lemma 4.2] that if S is a t-splitting set, then a prime t-ideal Q of D that intersects S, intersects S in detail i.e., every nonzero prime ideal contained in Q also intersects S. Note also that if P is a prime t-ideal of D such that $P \cap S \neq \emptyset$, then for every $0 \neq x \in P$, we have $xD = (AB)_t$, where A and B are integral ideals of D such that $(A, s)_t = D$ for all $s \in S$ and $B \cap S \neq \emptyset$, which forces $A \nsubseteq P$. This is because $(A, s)_t = D$ for all $s \in S$ and so for $s \in P \cap S$.

We begin this section by studying an integral domain in which every saturated multiplicative subset is a t-lcm splitting set. Recall that a WFD is a *generalized UFD* if every pair of non v-coprime primary elements is comparable (i.e., one of the two divides the other). For more on generalized UFDs, see [5].

Proposition 3.1. *Let D be an integral domain. Then the following statements are equivalent.*

(1) *Every saturated multiplicative subset of D is a t-lcm splitting set.*
(2) *Every saturated multiplicative subset of D is a lcm splitting set.*
(3) *D is a weakly factorial PVMD.*
(4) *D is a weakly factorial GCD-domain.*
(5) *D is a GCD generalized Krull domain.*
(6) *D is a generalized UFD.*

Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) follow from the following facts that every saturated multiplicative subset of D is a splitting set if and only if D is a WFD, if and only if D is a weakly Krull domain and $Cl(D) = 0$ [10, Theorem]; that a PVMD D is a GCD-domain if and only if $Cl(D) = 0$ [15, Proposition 2]; and that $D \setminus \{0\}$ is a t-lcm (resp., lcm) splitting set if and only if D is a PVMD (resp., GCD-domain). For (4) $\Leftrightarrow$ (6), see [5, Theorem 7]. □

Let S be a splitting set of an integral domain D. It is well known that S is an lcm splitting set if and only if D_T is a GCD-domain, where T is the m-complement of S [3, Proposition 2.4]. We would like to prove a t-lcm t-splitting set analog of this result (Theorem 3.4). For this some preparation seems to us necessary.

Lemma 3.2. *Let D be a quasilocal domain with maximal ideal M. If there is a nonzero element $\alpha \in M$ such that for all $x \in M$, (α, x) is principal, then D is a t-local domain.*

Proof. We first show that if h is a nonunit factor of α, then (h, x) is also a (proper) principal ideal of D. For this, let $b = \frac{\alpha}{h}$. Then $b(h, x) = (\alpha, bx)$ which is principal by the stated property of α, and hence $\frac{1}{b}(\alpha, bx) = (h, x)$ is principal. Now consider $(\alpha, x_1, x_2, \ldots, x_n)$ for any $x_1, \ldots, x_n \in M$. Then as $(\alpha, x_1, x_2, \ldots, x_n) = ((\alpha, x_1), x_2, \ldots, x_n) = (h_1, x_2, \ldots, x_n)$ where $h_1 | \alpha$, we can carry out the procedure to conclude that $(\alpha, x_1, x_2, \ldots, x_n)$ is principal. Now as $(\alpha, x_1, x_2, \ldots, x_n)$ is principal, $(\alpha, x_1, x_2, \ldots, x_n)_t = (\alpha, x_1, x_2, \ldots, x_n) \subseteq M$; hence $(x_1, x_2, \ldots, x_n)_t \subseteq (\alpha, x_1, x_2, \ldots, x_n)_t \subseteq M$ for all $x_1, x_2, \ldots, x_n \in M$. □

This result has an immediate consequence.

Lemma 3.3. *Let S be a t-lcm t-splitting set of an integral domain D. Then for every maximal t-ideal P of D with $P \cap S \neq \emptyset$, D_P is t-local.*

Proof. Let $s \in P \cap S$. Then (s, x) is t-invertible for every $x \in P$ since S is a t-lcm t-splitting, and hence $(s, x)D_P$ is principal for every $x \in PD_P$. Thus D_P is t-local by Lemma 3.2. □

Theorem 3.4. *Let D be an integral domain with quotient field K, S a t-splitting set of D, and $\mathcal{T} = \{A_1 \cdots A_n | A_i = d_i D_S \cap D$ for some $0 \neq d_i \in D\}$. Then the following statements are equivalent.*

1. *S is a t-lcm t-splitting set.*
2. *Every finite type integral v-ideal of D intersecting S is t-invertible.*
3. *$D_{\mathcal{T}} = \{x \in K | xC \subseteq D$ for some $C \in \mathcal{T}\}$ is a PVMD.*

Proof. (1) ⇒ (2) Let A be a finite type integral v-ideal of D such that $A \cap S \neq \emptyset$, and let P be a maximal t-ideal of D. If P contains A, then $P \cap S \neq \emptyset$; hence AD_P is principal by the proofs of Lemmas 3.2 and 3.3. If P does not contain A, then $AD_P = D_P$. Thus A is t-locally principal, and hence A is t-invertible [26, Corollary 2.7].

(2) ⇒ (1) Let $s \in S$ and $0 \neq d \in D$. Then $(s, d)_v$ is a finite type integral v-ideal of D intersecting S; so $(s, d)_v$ is t-invertible. Thus $(s, d)^{-1} = \frac{1}{sd}(sD \cap dD)$, and hence $sD \cap dD$, is t-invertible.

(2) ⇒ (3) Let P be a prime t-ideal of D such that $P \cap S \neq \emptyset$. We first show that D_P is a valuation domain. Let $0 \neq x, y \in PD_P$. We can assume that $x, y \in P$. Now as S is a t-splitting set, we can write $xD = (A_1B_1)_t$ and $yD = (A_2B_2)_t$, where A_i are t-coprime with every member of S and B_j intersect S, and hence A_i are not contained in P. Next by [6, Lemma 4.5], we have $(x, y)_t = (A_1B_1, A_2B_2)_t = ((A_1, A_2)(B_1, B_2))_t$. Now (B_1, B_2) is contained in P and intersects S, and so $(B_1, B_2)D_P$ is principal (see the proof of (1) ⇒ (2) above). Also, since $(A_1, A_2) \nsubseteq P$ we have that $(A_1, A_2)(B_1, B_2)D_P$ is principal, and thus $((x, y)D_P)_t = ((x, y)_t D_P)_t = (((A_1, A_2)(B_1, B_2))_t D_P)_t = ((A_1, A_2)(B_1, B_2)D_P)_t$ $= (A_1, A_2)(B_1, B_2)D_P$ is principal (see [26, Lemma 3.4] for the first and the third equalities). Moreover, since D_P is t-local by Lemma 3.3, we have that $(x, y)D_P$ is invertible. Thus every two generated ideal of D_P is invertible, and hence D_P is a valuation domain [21, Theorem 22.1].

Let Q be a maximal t-ideal of $D_{\mathcal{T}}$ and $P = Q \cap D$. Then P is a prime t-ideal of D such that $P \cap S \neq \emptyset$ (cf. [6, Theorems 4.3 and 4.10]); hence D_P is a valuation domain by the above paragraph. Also, since $D_P \subseteq (D_{\mathcal{T}})_Q$, $(D_{\mathcal{T}})_Q$ is a valuation domain [21, Theorem 17.6]. Thus $D_{\mathcal{T}}$ is a PVMD [23, Theorem 5].

(3) $\Rightarrow$ (2) Let A be a finite type integral v-ideal of D such that $A \cap S \neq \emptyset$, and let P be a maximal t-ideal of D. If $P \cap S = \emptyset$, then $A \nsubseteq P$; so $AA^{-1} \nsubseteq P$. Now assume that $P \cap S \neq \emptyset$. Then $(PD_{\mathcal{T}})_t \subsetneq D_{\mathcal{T}}$ (cf. [6, Theorem 4.10]). Let Q be a prime ideal of $D_{\mathcal{T}}$ minimal over $(PD_{\mathcal{T}})_t$ such that $Q \subseteq PD_P \cap D_{\mathcal{T}}$. Then Q is a t-ideal of $D_{\mathcal{T}}$ such that $Q \cap D = P$, and hence $D_P \subseteq (D_{\mathcal{T}})_Q \subseteq (D_P)_{PD_P} = D_P$; so $D_P = (D_{\mathcal{T}})_Q$. Since $D_{\mathcal{T}}$ is a PVMD and Q is a t-ideal, D_P is a valuation domain [23, Theorem 5]. Hence $D_P = (AD_P)(AD_P)^{-1} = (AD_P)(A^{-1}D_P) = (AA^{-1})D_P$; so $AA^{-1} \nsubseteq P$ (see [26, Lemma 3.4] for the second equality). Therefore $(AA^{-1})_t = D$; hence A is t-invertible. □

The following corollary appears in the proof of (2) $\Rightarrow$ (3) of Theorem 3.4. This also strengthens Lemma 3.3 since a valuation domain is t-local.

Corollary 3.5. *Let S be a t-lcm t-splitting set of an integral domain D, and let P be a maximal t-ideal of D intersecting S. Then D_P is a valuation domain.*

Corollary 3.6. *Let S be a t-lcm t-splitting set of an integral domain D. Then D is a PVMD (resp., UMT-domain) if D_S is a PVMD (resp., UMT-domain).*

Proof. Assume that D_S is a PVMD (resp., UMT-domain), and let P be a maximal t-ideal of D. CASE 1. $P \cap S = \emptyset$. Then PD_S is a prime t-ideal of D_S [6, Theorem 4.9], and hence $D_P = (D_S)_{PD_S}$ is a valuation domain [23, Theorem 5] (resp., the integral closure of $D_P = (D_S)_{PD_S}$ is a Prüfer domain [18, Theorem 1.5]). CASE 2. $P \cap S \neq \emptyset$. Then D_P is a valuation domain by Corollary 3.5. Thus D is a PVMD [23, Theorem 5] (resp., a UMT-domain [18, Theorem 1.5]). □

Let D be an integral domain with quotient field K and let X be an indeterminate over D. Let $G = \{f \in D[X] | (A_f)_v = D\}$. This set was first used by Gilmer [20], who showed that if D is a v-domain, then D is a PVMD if and only if $D[X]_G$ is a Bezout domain. (Recall that D is a v-domain if every finitely generated ideal I of D is v-invertible, i.e., $(II^{-1})_v = D$.) The v-domain condition was relaxed first to integrally closed integral domains by Zafrullah [36] and then shown to be completely unnecessary by Kang [26]. Huckaba and Papick [25] have also used this set in the more general setting of rings with zero divisors and so has Kang [28]. We note that if we show that G is a t-lcm t-splitting set of $D[X]$, then we have actually shown that if $D[X]_G$ is a PVMD, then D is a PVMD.

Proposition 3.7. *Let D be an integral domain, X an indeterminate over D, $G = \{f \in D[X] | (A_f)_v = D\}$, and $\emptyset \neq S \subseteq G$ a multiplicative subset of $D[X]$. Then S is a t-lcm t-complemented t-splitting set of $D[X]$. In particular, if N is the t-complement of S, then $D[X]_N$ is a PID.*

Proof. Let $\mathcal{P}$ be the set of all uppers to zero in $D[X]$ that intersect S. Then by [24, Theorem 1.4], every member of $\mathcal{P}$ is a maximal t-ideal and t-invertible. Thus $(*)$ if A is an integral ideal of $D[X]$ such that A is not contained in any member of $\mathcal{P}$, then $(A, g)_t = D[X]$ for all $g \in S$. Indeed, (A, g) is contained in no upper to zero, so it contains some nonzero $a \in D$. Consequently, $(A, g)_t$ contains $(a, g)_t = D[X]$.

Now let $0 \neq f \in D[X]$. If f does not belong to any member of $\mathcal{P}$, then by $(*)$ above $(f, g)_t = D[X]$ for all $g \in S$, and so we can write $fD[X] = (fD[X])D[X] = ((fD[X])D[X])_t$, where obviously $D[X]$ is t-invertible such that $D[X] \cap S \neq \emptyset$. Next let f be in some members of $\mathcal{P}$. Then, as f belongs to only a finite number of uppers to zero in $D[X]$, f belongs to only finitely many members of $\mathcal{P}$. Suppose that $f \in P_1, \dots, P_r$ only. Let us start with P_1. Since P_1 is t-invertible and $f \in P_1$, we have $fD[X] = (A_1 P_1)_t$ where $A_1 = fP_1^{-1}$. If $A_1 \nsubseteq P_1$, we are done and move to the next stage. Yet if $A_1 \subseteq P_1$, we can write $(A_1)_t = (A_2 P_1)_t$ and so $fD[X] = (A_2(P_1)^2)_t$, and since $D[X]_{P_1}$ is a DVR, continuing this way we come to a stage where $fD[X] = (A_{n_1}(P_1)^{n_1})_t$ and $A_{n_1} \nsubseteq P_1$. Repeating this procedure with $P_2, \dots, P_n$, we conclude that $fD[X] = (AP_1^{n_1} \cdots P_r^{n_r})_t$ where A_t and hence A is not contained in any member of $\mathcal{P}$. For if A were in any member P of $\mathcal{P}$, then $P \neq P_i$ and P would contain A_t which contains f, and this would contradict the assumed fact that $f \in P_1, \dots, P_r$ only. Now by $(*)$, $(A, g)_t = D[X]$ for all $g \in S$. Finally since $(P_1^{n_1} \cdots P_r^{n_r})_t = B$ intersects S, S is a t-splitting set.

Let K be the quotient field of D, $\mathcal{T} = \{A_1 \cdots A_n | A_i = f_i D[X]_S \cap D$ for some $0 \neq f_i \in D[X]\}$, and $D[X]_{\mathcal{T}} = \{x \in K(X) | xC \subseteq D[X]$ for some $C \in \mathcal{T}\}$. Then $K[X] \subseteq D[X]_{\mathcal{T}}$ since $dD[X]_S \cap D[X] = dD[X]$ for all $0 \neq d \in D$. Also, since $K[X]$ is a PID, $D[X]_{\mathcal{T}} = K[X]_T$ for some saturated multiplicative subset T of $K[X]$ [21, Proposition 27.3]. Let $N = T \cap D[X]$; then clearly $D[X]_{\mathcal{T}} = D[X]_N$, and thus S is a t-lcm t-complemented t-splitting set by Theorem 3.4. □

Recall that an integral domain D is a *Mori domain* if D satisfies the ascending chain condition on integral divisorial ideals. The most important examples of Mori domains include Noetherian domains and Krull domains. Also, recall that D is a *P-domain* if for every associated prime ideal P of D, D_P is a valuation domain. Clearly PVMDs are P-domains since an associated prime ideal is a t-ideal [23, Theorem 5]. But a P-domain need not be a PVMD (see [31]).

Corollary 3.8. *Let D be an integral domain, X an indeterminate over D, $G = \{f \in D[X] | (A_f)_v = D\}$, and $\emptyset \neq S \subseteq G$ a multiplicative subset of $D[X]$. Then $D[X]$ is a Krull domain (resp., Mori domain, integrally closed, completely integrally closed, P-domain, UMT-domain, PVMD) if $D[X]_S$ is a Krull domain (resp., Mori domain, integrally closed, completely integrally closed, P-domain, UMT-domain, PVMD).*

Proof. We first note that S is a t-complemented t-splitting set by Proposition 3.7. Let T be the t-complement of S. Then $D[X]_T$ is a PID by Proposition 3.7 and $D[X] = D[X]_S \cap D[X]_T$ [6, Theorem 4.3]. Thus $D[X]$ is a Krull domain (resp., Mori domain, integrally closed, completely integrally closed) by [19, Proposition 1.4] (resp., [32, Théorème 1], [29, Theorem 52], [21, Ex. 11, page 145]).

Suppose that $D[X]_S$ is a P-domain, and let P be an associated prime ideal of $D[X]$. Then P is minimal over an ideal of the form $(fD[X] : gD[X])$ for some $0 \neq f, g \in D[X]$; so P is a t-ideal. Since S is a t-splitting set, either $P \cap S = \emptyset$ or $P \cap T = \emptyset$ (cf. [6, Theorem 4.10]); hence $PD[X]_{S'}$ is minimal over $(fD[X]_{S'} : gD[X]_{S'}) \subsetneq D[X]_{S'}$ [21, Theorem 4.4], where $S' = S$ or T. Also, since $D[X]_{S'}$ is a P-domain in any case, we have that $D[X]_P = (D[X]_{S'})_{PD[X]_{S'}}$ is a valuation domain, and thus $D[X]$ is a P-domain. The UMT-domain and PVMD cases follow directly from Corollary 3.6 and Proposition 3.7. □

In [11], it was shown that an integral domain D is integrally closed if and only if every irreducible monic polynomial over D is a principal prime. Recently, McAdam [30] has shown that D is integrally closed if and only if every (non constant) monic polynomial over D is uniquely expressible as a product of irreducible (monic) polynomials. Let us see how to put to use this windfall of information, i.e., the fact that if D is integrally closed then every irreducible (non constant) monic polynomial in $D[X]$ is a principal prime. Our application is for the ring $D\langle X\rangle = D[X]_U$, where U is the multiplicative subset of $D[X]$ generated by monic polynomials in $D[X]$. This ring was used by Quillen [33] in solving the Serre conjecture and it is considered worth-while to check the transfer of properties from D to $D\langle X\rangle$ and vice versa. This construction has been studied by several authors including D.D. Anderson, D.F. Anderson, and R. Markanda [2] and Le Riche [34]. The properties we have in mind are of interest in that they are the most basic and other properties can be derived from these. We also provide more direct proofs of other results via Nagata's theorem for UFD's and its modification for Krull domains.

Let us recall that an element $0 \neq x \in D$ is a *primal element* if for all $0 \neq a, b \in D$, $x|ab$ implies that $x = rs$ where $r|a$ and $s|b$. A primal element x is said to be *completely primal* if every factor of x is primal. An integrally closed integral domain D is called *Schreier* if every nonzero element of D is primal. The notion of a Schreier domain was introduced by Paul Cohn [17]. In [17], the following statements were established: (1) Let D be an integrally closed integral domain and S a multiplicative subset of D. Then (i) if D is Schreier, so is D_S and (ii) if D_S is Schreier and S is generated by completely primal elements of D, then D is Schreier [17, Theorem 2.6]. (2) Let D be a Schreier domain and X an indeterminate over D, then $D[X]$ is again a Schreier domain [17, Theorem 2.7]. Indeed, if $D[X]$ is Schreier, then D is also Schreier since $D\backslash\{0\}$ is a saturated multiplicative subset of $D[X]$ consisting of completely primal elements of $D[X]$ and hence of D, via degree considerations, and further since $D[X]$ being integrally closed implies D integrally closed.

Proposition 3.9. *Let D be an integral domain, X an indeterminate over D, and let U be the set of all monic polynomials in $D[X]$. Then D is a Schreier domain if and only if $D[X]_U = D\langle X\rangle$ is a Schreier domain.*

Proof. Assume that $D[X]_U = D\langle X\rangle$ is a Schreier domain. Then $D[X]_U$ is integrally closed, and hence $D[X]$ is integrally closed by Corollary 3.8. This means that U is generated by principal primes of $D[X]$ (cf. [11, Theorem 3.2] or [30, Theorem]). Hence U is a splitting set of $D[X]$ generated by principal primes (Corollary 2.5 and Proposition 3.7), and thus $D[X]$ is Schreier by [17, Theorem 2.6] mentioned above. Now $D[X]$ being Schreier implies that D is Schreier as we remarked above. The converse is obvious via Cohn's results [17, Theorems 2.6 and 2.7] mentioned above. □

Now we attend to another basic property, that of being a PVMD.

Proposition 3.10. *Let D be an integral domain, X an indeterminate over D, and let U be the set of all monic polynomials in $D[X]$. Then D is a PVMD if and only if $D[X]_U = D\langle X\rangle$ is a PVMD.*

Proof. Since U is a subset of $G = \{f \in D[X] | (A_f)_v = D\}$, it follows directly from Corollary 3.8 that if $D[X]_U$ is a PVMD then $D[X]$ is a PVMD, and thus D is a PVMD (cf. [8, Corollary 3.3 (3)] or [26, Theorem 3.7]). The converse follows from the well-known facts that if D is a PVMD then $D[X]$ is a PVMD [26, Theorem 3.7] and that rings of fractions of PVMDs are PVMDs. □

The fun starts when we note that D is a GCD-domain if and only if D is a PVMD and Schreier [37, Theorem 3.6]. So, combining Propositions 3.9 and 3.10 we have the following result due to Le Riche [34, Proposition 1.1]

Proposition 3.11. *Let D be an integral domain, X an indeterminate over D, and let U be the set of all monic polynomials in $D[X]$. Then D is a GCD-domain if and only if $D[X]_U = D\langle X\rangle$ is a GCD-domain.*

Proposition 3.11 can be proved more directly with a very short proof via a Nagata-type Theorem: If S is an lcm-splitting set of D such that D_S is a GCD-domain, then D is a GCD-domain (cf. [3, Theorem 4.3]). But then we would miss Propositions 3.9 and 3.10 completely.

Proposition 3.12. ([2, Theorems 5.2 and 5.3] and [34, Proposition 1.2]) *Let D be an integral domain, X an indeterminate over D, and let U be the set of all monic polynomials in $D[X]$. Then D is a Krull domain (resp., UFD) if and only if $D[X]_U = D\langle X\rangle$ is a Krull domain (resp., UFD).*

Proof. Clearly if D is a Krull domain (resp., UFD), then so is $D\langle X\rangle$. For the converse, we note that by Corollay 3.8, D is a Krull domain anyway. For the UFD part, we note that D is a UFD if and only if D is Krull and Schreier, and apply Proposition 3.9. Or note that $D\langle X\rangle$ UFD implies that D is integrally closed and so U is generated by principal primes, and then apply Nagata's theorem for UFDs to conclude that $D[X]$ is a UFD, which forces D to be a UFD. □

Recall that an integral domain D is an *almost GCD-domain* (AGCD-domain) if for each $0 \neq a, b \in D$, there is an integer $n = n(a, b) \geq 1$ such that $a^n D \cap b^n D$ is principal. It is well known that if D is an AGCD-domain then $Cl(D)$ is torsion and that D is an integrally closed AGCD-domain if and only if D is a PVMD with $Cl(D)$ torsion.

Corollary 3.13. *Let D be an integral domain, X an indeterminate over D, and let U be the set of all monic polynomials in $D[X]$. Then D is an integrally closed*

AGCD-domain (resp., almost factorial Krull domain) if and only if $D[X]_U = D\langle X\rangle$ *is an integrally closed AGCD-domain (resp., almost factorial Krull domain).*

Proof. Recall that if D is integrally closed then $Cl(D) = Cl(D\langle X\rangle)$ [14, Corollary 1.3], and thus the results follow from Propositions 3.10 and 3.12. □

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