14
t-Invertibility and Comparability

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Let D be an integral domain with quotient field K and let $F(D)$ denote the set of nonzero fractional ideals of D. For $A \in F(D)$ we define $A^{-1} = \{x \in K | xA \subseteq D\}$. An ideal $A \in F(D)$ is t-invertible if (i) $(AA^{-1})^{-1} = D$, (ii) there exists $x_1, x_2, \ldots, x_n \in A$ such that $(x_1, \ldots, x_n)^{-1} = A^{-1}$, and (iii) $A^{-1} = ((y_1, \ldots, y_m)^{-1})^{-1}$ for some $y_1, \ldots, y_m \in K$. The notion of t-invertibility has been useful in characterizing Krull domains in various ways [HZ], [J], [K] and [MZ].

Houston and Zafrullah in [HZ] discussed situations in which t-invertibility arises naturally. They showed that if a nonzero integral ideal A of a polynomial ring $D[X]$ contains a polynomial f with $A_f^{-1} = D$, then A is t-invertible. Here A_f denotes the content of $f(X)$. This result led us to ask, in a general setting, what nonzero elements x of a domain D are *t-invertibility elements* of D in the sense that each ideal of D containing x is t-invertible. We obtain a characterization of such elements (and more generally, of the so-called t-invertibility ideals of D; see Section 1 for the definition) in Theorem 1.3. The proof of (1.3) uses some of the following terminology and results; our main references are [G, Sections 32, 34] and [AA].

I. A star operation is a function $F \rightarrow F^*$ from $F(D)$ to $F(D)$ with the following properties: if $A, B \in F(D)$ and $a \in K \setminus \{0\}$, then
 (i) $(a)^* = (a)$ and $(aA)^* = aA^*$.
 (ii) $A \subseteq A^*$ and if $A \subseteq B$, then $A^* \subseteq B^*$.
 (iii) $(A^*)^* = A^*$.

The mapping $A \rightarrow (A^{-1})^{-1} = A_v$ on $F(D)$ is a star operation called the *v-operation* on D. Similarly, $A \rightarrow A_t = \cup F_v$, where F ranges over the set of finitely generated

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nonzero subideals of A, is a star operation called the *t-operation* on D. The identity mapping $F \to F$ on $F(D)$ is a star operation called the *d-operation* on D. A star operation $F \to F^*$ is said to be *of finite character* if $A^* = \cup_i A_i^*$ for each $A \in F(D)$, where $\{A_i\}$ is the family of nonzero finitely generated fractional ideals of D contained in A. Given any star operation $F \to F^*$ on D, the function $A \to A^{*_s} = \cup_i A_i^*$, where $\{A_i\}$ is the family of nonzero finitely generated fractional ideals of D contained in A, is a star operation of finite character on D, and $B^* = B^{*_s}$ for each finitely generated $B \in F(D)$. The d-operation on D is of finite character, and the t-operation is the v_s-operation.

II. For all $A, B \in F(D)$ and for each star operation $\star$, $(AB)^\star = (AB^\star)^\star = (A^\star B^\star)^\star$, $(A^{-1})^\star = A^{-1}$, and $(A^\star)_v = A_v = (A_v)^\star$.

III. An ideal A of D is said to be a $\star$-ideal, for a given star operation $\star$, if $A^\star = A$. Thus from II above $A^\star$, A^{-1} and A_v are $\star$-ideals. Likewise we define v-ideals and t-ideals and note that $A^\star \subseteq A_v$ for any star operation $\star$.

IV. An integral ideal A of D is said to be a maximal $\star$-ideal, for a given star operation $\star$, if $A \neq D$ and A is maximal with respect to being a $\star$-ideal. If $\star$ is of finite character, then every proper integral $\star$-ideal is contained in a maximal $\star$-ideal. A maximal $\star$-ideal is necessarily a prime ideal [J, p. 30].

V. Let $\{D_i\}_{i \in I}$ be a family of overrings of D such that $D = \bigcap_{i \in I} D_i$. Then the operation on $F(D)$ defined by $A \mapsto \cap AD_i$ is a star operation. This star operation is said to be *induced by* $\{D_i\}$ [G], [A].

VI. An ideal $A \in F(D)$ is *$\star$-invertible* if there exists $B \in F(D)$ such that $(AB)^\star = D$. In this case, $B^\star = A^{-1}$. This yields the definitions of v-invertibility and t-invertibility of A. The definition of the t-operation implies that if A is t-invertible, then there exists a finitely generated ideal B contained in AA^{-1} such that $B_t = B_v = D$; hence the definition of t-invertibility in the first paragraph.

In Section 2 we consider the notion of a *comparable element* of a domain D, as defined by Anderson and Zafrullah in [AZ2] (the definition is that $d \in D\backslash(0)$ is comparable if (d) compares with each ideal of D under inclusion). Theorem 2.3 characterizes domains that contain a nonunit comparable element. In Section 3 we relate the concepts of t-invertibility and t-local comparability.

1. A CHARACTERIZATION OF t-INVERTIBILITY ELEMENTS AND IDEALS

Our main goal in this section is to characterize elements or ideals of a domain D such that each ideal of D in which they are contained is t-invertible. We achieve this goal in Theorem 1.3. Our first result deals, however, with ordinary invertibility of ideals of a commutative ring.

THEOREM 1.1. *Suppose A is an ideal of the commutative unitary ring R. Each ideal of R containing A is invertible if and only if A is a finite product of invertible maximal ideals of R.*

Proof. Suppose each ideal of R containing A is invertible. Since invertible ideals are finitely generated, R/A is Noetherian. Thus if some ideal of R containing A is not a finite product of maximal ideals, there is an ideal B maximal with respect to this property. Then B is not maximal, so let M be a maximal ideal of R containing B. Since M is invertible, $B = MC$ for some ideal $C \supseteq B$. Because B is invertible and $M \neq R$, we have $B < C$. Hence C is a finite product of maximal ideals, and so is B. This contradiction shows that each ideal of R containing A is a finite product of invertible maximal ideals, and in particular, A has this property.

Conversely, if $A = M_1 M_2 \cdots M_k$ is a finite product of invertible maximal ideals M_i, then the ideals M_i are the only prime ideals of R that contain A. Hence each prime ideal of R containing A is invertible, and this implies, in the usual fashion (cf. [Kp, Exer. 10, p. 11]), that each ideal of R containing A is invertible. ∎

REMARK 1.2. If the conditions of Theorem 1.1 are satisfied, then A is invertible and we can see that R/A is a zero-dimensional PIR by the following argument: That R/A is zero-dimensional with only finitely many maximal ideals is clear; to see that R/A is a PIR, it suffices to show that M/A is principal for each maximal ideal M of R containing A. Let $\{M_i\}_{i=1}^n$ be the (finite) set of maximal ideals of R distinct from M. Choose $m \in M - [M^2 \cup (\cup_{i=1}^n M_i)]$. Then (A, m) is invertible, so $(A, m) = MB$ for some ideal B of R containing A. Moreover, B is contained in neither M nor any M_i by choice of m. Hence $B = R$, $M = (A, m)$, and M/A is principal as we wished to show.

The converse of the observation just made fails. For example, if t is an indeterminate over the field K and if $R = K[t^2, t^3]$, then $A = t^2 R$ is invertible and $R/A \simeq K[X]/(X^2)$ is a zero-dimensional PIR. However, the ideal (t^2, t^3) of R contains A, but is not invertible.

The proof of Theorem 1.1 can be adapted to give a characterization of t-invertibility elements, as indicated in Theorem 1.3.

THEOREM 1.3. *Suppose A is a nonzero proper ideal of the domain D. Each ideal of D containing A is t-invertible if and only if A_t is a finite t-product of maximal t-ideals of D, each of which is t-invertible.*

Proof. Suppose each ideal of D containing A is t-invertible. We show that each t-ideal B of D containing A is a finite t-product of maximal t-ideals, each of which is t-invertible. Because the t-operation is of finite character, a t-invertible t-ideal is of finite type [J, p. 30], [Kr], so each ideal of D containing A is of finite t-type. Moreover, the union of a chain of t-ideals is again a t-ideal. It then follows by the usual argument that the set of t-ideals of D containing A is inductive under $\subseteq$. Hence, if there exists a t-ideal B of D containing A such that B is not a finite t-product of maximal t-ideals, there is a maximal such B. Then B is not a maximal t-ideal. Let M be a maximal t-ideal containing B. If $C = BM^{-1}$, then C is an integral ideal of D containing B, and because M is t-invertible, $B = (MC)_t$. We cannot have $B = C$, for t-invertibility of B would then imply that $M = M_t = D$. Hence C_t is a t-product of maximal t-ideals, as is B. This contradiction establishes the desired assertion, and in particular, A is a finite product of t-invertible maximal t-ideals.

Conversely, if $A_t = (M_1 M_2 \cdots M_k)_t$ is a finite t-product of t-invertible maximal t-ideals M_i, then since $M_1 \cdots M_k \subseteq A_t$, the ideals M_i are the only proper prime t-ideals of D that contain A_t. Hence each prime t-ideal of D containing A is t-invertible, and this implies [MZ, Prop. 2.1] that each ideal of D containing A is t-invertible. ∎

An examination of the proof of Theorem 1.3 reveals that the only special property of the t-operation, among star operations, used in the proof is that the t-operation is of finite character. Hence Theorem 1.3 generalizes to the following result.

THEOREM 1.3′. *Suppose $F \to F^*$ is a star operation of finite character on the domain D. Let A be a nonzero proper ideal of D. Each ideal of D containing A is $\star$-invertible if and only if A is a finite $\star$-product of $\star$-invertible maximal $\star$-ideals.*

Since the t-operation and the d-operation on a domain D are of finite character, Theorem 1.3 and the case of Theorem 1.1 where R is an integral domain could be obtained from Theorem 1.3′. Theorem 1.1 itself seems, however, not to be a direct corollary of Theorem 1.3′.

Suppose $F \to F^*$ is a star operation of finite character on D. We call a nonzero ideal A of D a *$\star$-invertibility ideal* if either $A^* = D$ or else A satisfies the equivalent conditions of Theorem 1.3′; an element d of D is a *$\star$-invertibility element* if (d) is a $\star$-invertibility ideal. Theorem 1.3′ gives rise to the following corollary.

COROLLARY 1.4. *Let the notation and hypothesis be as in the preceding paragraph and let S be the set of $\star$-invertibility ideals of D. Then S is closed under multiplication of ideals and under taking overideals within D. In particular, the set S of $\star$-invertibility elements of D is a saturated multiplicative system in D.*

Proof. Theorem 1.3′ implies that S is closed under multiplication, and it follows from the proof of Theorem 1.3 that if $A \in S$ and if B is an overideal of A in D, then $B \in S$. ∎

REMARK 1.5. If $F \to F^*$ is a $\star$-operation on D of finite character, the proof of Theorem 1.3 can easily be adapted to show that a nonzero ideal A of D is a $\star$-invertibility ideal if and only if A satisfies the following three conditions: (i) A belongs to only finitely many maximal t-ideals $M_1, \ldots, M_s$; (ii) each M_i is t-invertible; (iii) each M_i is minimal in the set of prime t-ideals of D containing A. From this characterization and from Corollary 1.4 it follows that if S is the set of $\star$-invertibility ideals of D and if A and B are ideals of D, then (a) $\mathrm{rad}(A) \in S$ if and only if $A \in S$, and (b) $A \cap B \in S$ if and only if $A, B \in S$.

Houston and Zafrullah [HZ] proved a special case of the "only if" part of Theorem 1.3. To wit, let X be an indeterminate over D and let $f(X) \in D[X]$ be such that $A_f^{-1} = D$. Then according to [HZ]:

1. If A is an (integral) ideal of $D[X]$ with $f(X) \in A$, then A is t-invertible.
2. Every prime upper to (0) in $D[X]$ that contains $f(X)$ is a t-invertible t-ideal, and hence a maximal t-ideal.
3. Using (1), (2), and some localization, it follows that $(f(X)) = P_1^{(e_1)} \cap \ldots \cap P_r^{(e_r)} = (P_1^{e_1} \cdots P_r^{e_r})_t$, where $\{P_1, \ldots, P_r\}$ is the set of prime uppers to (0) that contain $f(X)$.

From these observations we conclude:

(A) If $(A_f)_v = D$, then $(f(X))$ is a t-product of primes.

(B) If $(A_f)_v = D$, then $f(X)$ belongs to a finite number of maximal t-ideals $P_1, P_2, \ldots, P_r$, and for each of these, $D[X]_{P_i}$ is a discrete rank-one valuation domain. (Because the ideals P_i are prime uppers to (0).)

In our terminology, (1) merely says that f is a t-invertibility element of $D[X]$ if $A_f^{-1} = D$. We proceed to show, in an abstract setting, that an element of a domain satisfying conditions analogous to those of (B) is a t-invertibility element. For the sake of brevity, we say that a nonzero element d of a domain D is a *t-element* of D if either d is a unit of D, or else d belongs to only finitely many maximal t-ideals $P_1, \ldots, P_r$ of D and each D_{P_i} is a rank-one discrete valuation domain. We show in Theorem 1.6 that t-elements are t-invertibility elements. Recall that an ideal A of D is *strictly v-finite* if there exists a finitely generated ideal $B \subseteq A$ such that $A_v = B_v$.

THEOREM 1.6. *If d is a t-element of the domain D, then d is a t-invertibility element of D.*

Proof. The statement is clear if d is a unit of D. If d is a nonunit, we show that each ideal A of D containing d is t-invertible. Thus, let $P_1, \ldots, P_r$ be the maximal t-ideals of D containing d. If $\{P_\alpha\}$ is the set of all maximal t-ideals of D, then $D = \bigcap_\alpha D_{P_\alpha}$ [Gr, Prop. 4], and AD_{P_i} is principal for $1 \le i \le r$. Hence Lemma 1.10 of [MMZ] shows that A is strictly v-finite, and [MMZ, Cor. 1.16] then implies that A is t-invertible. ∎

The converse of Theorem 1.6 fails. For example, in the classical $D + M$ construction $\mathbb{Z} + X\mathbb{Q}[[X]]$, 2 is a t-invertibility element, but not a t-element. The problem in this case is that the unique maximal t-ideal containing 2 has height greater than 1. We show in Theorem 1.9 that a t-invertibility element without this defect is a t-element.

LEMMA 1.7. *Suppose A is a t-invertible ideal of the domain D. If S is a multiplicative system in D, then AD_S is t-invertible in D_S.*

Proof. Both A and A^{-1} are of finite type — say $A_t = B_t$ and $(A^{-1})_t = C_t$, where B and C are finitely generated subideals of A and A^{-1}, respectively. Then by Lemma 4 of [Z], $(BCD_S)_t = (BCD_S)_v = ((BC)_vD_S)_v = (DD_S)_v = D_S$. Hence BD_S is t-invertible. Again using [Z, Lemma 4], we have $(AD_S)_t \subseteq (A_tD_S)_t = (B_tD_S)_t = (BD_S)_t \subseteq (AD_S)_t$. Therefore $(AD_S)_t = (BD_S)_t$ and AD_S is t-invertible, as asserted. ∎

LEMMA 1.8. *Let (D, M) be a one-dimensional quasilocal domain. If M is t-invertible, then D is a DVR.*

Proof. Because M is t-invertible, it is of finite type. Also, M is the unique maximal t-ideal of D. Corollary 1.6 of [MMZ] then implies that $MD_M = M$ is principal, so D is a DVR, as we wished to show. ∎

THEOREM 1.9. *Suppose d is a t-invertibility element of the domain D such that $(d) = (P_1P_2 \cdots P_r)_t$, where each P_i is a maximal t-ideal of D of height one. Then d is a t-element of D.*

Proof. It is clear that $P_1, \ldots, P_r$ are the only maximal t-ideals of D containing d. To see that each D_{P_i} is a DVR, we observe that P_i is t-invertible since $(P_1 \cdots P_r)_t = (d)$ is t-invertible. Lemma 1.7 shows that $P_iD_{P_i}$ is t-invertible in D_{P_i}, and Lemma 1.8 then implies that D_{P_i} is a DVR. ∎

To return briefly to results of [HZ], we remark that while each $f \in D[X]$ such that $A_f^{-1} = D$ is a t-invertibility element of $D[X]$, there may exist others. For example, if $D[X]$ is a Krull domain (that is, if D is a Krull domain), then each nonzero element of $D[X]$ is a t-invertibility element. (Conversely, if E is an integral domain with set $E - \{0\}$ of t-invertibility elements, then E is a Krull domain.)

2. COMPARABLE ELEMENTS

Suppose D is an integral domain. Recall that Anderson and Zafrullah in [AZ2] called a nonzero element d of D *comparable* if (d) compares with each ideal of D under inclusion. In Theorem 3.1 we establish a connection between the concepts of comparability and t-invertibility, but this section is devoted primarily to a determination of those domains D that admit a nonunit comparable element (Theorem 2.3). We record the following useful result of Anderson and Zafrullah that is established in the proof of Theorem 2 of [AZ2].

PROPOSITION 2.1. (Anderson-Zafrullah) *The set of comparable elements of an integral domain D is a saturated multiplicative system in D.*

Before proving Theorem 2.3, we state and prove Proposition 2.2. While this result should be known, we have been unable to locate a satisfactory reference in the literature (cf. [ABDFK, Lemma 2.1]).

PROPOSITION 2.2. *Suppose (D, M) is a quasilocal domain. Let $\varphi : D \to D/M$ be the canonical homomorphism, and let (J, P) be a quasilocal subring of D/M. Then $\varphi^{-1}(J) = R$ is quasilocal with maximal ideal $\varphi^{-1}(P)$.*

Proof. It is clear that $\varphi^{-1}(P)$ is a maximal ideal of R, that $M \subseteq \varphi^{-1}(P)$, and that $R \subseteq D$. We note that $1 + M \subseteq U(R)$, the set of units of R. To see this, let $x = 1 + m$, where $m \in M$; x is a unit of D and its inverse y is necessarily of the form $1+n$ for some $n \in M$ since $\varphi(x) = \varphi(1)$. Hence $y \in R$, and x is a unit of R, as asserted. To show $\varphi^{-1}(P)$ is the unique maximal ideal of R, take $r \in R - \varphi^{-1}(P)$. Then $\varphi(r) \in J - P$, so $\varphi(r)$ is a unit of J. Thus there exists $s \in R$ such that $\varphi(rs) = \varphi(1)$, so $rs \in 1 + M \subseteq U(R)$. Therefore $r \in U(R)$, as we wished to show. ∎

In the statement of Theorem 2.3, we call an ideal A of a ring R a *comparable ideal* of R if A compares with each ideal of R under inclusion.

THEOREM 2.3. *Suppose the integral domain D contains a nonzero nonunit comparable element; let Y be the set of nonzero comparable elements of D. Then:*

(1) $P = \cap\{(c) \mid c \in Y\}$ is a prime ideal of D, and $D \backslash P = Y$.
(2) D/P is a valuation domain.
(3) $P = PD_P$.
(4) D is quasilocal, P is a comparable ideal of D, and $\dim D = \dim(D/P) + \dim(D_P)$.

Moreover, if J is any integral domain such that there exists a nonmaximal prime ideal Q of J such that (a) J/Q is a valuation domain, and (b) $Q = QJ_Q$, then each element of $J \backslash Q$ is comparable. If, in addition, Q is minimal with respect to properties (a) and (b), then $J \backslash Q$ is the set of nonzero comparable elements of J.

Proof. To prove (1), choose $x, y \in D \backslash P$. There exist elements c and d of Y such that $x \notin (c)$ and $y \notin (d)$. Hence $(x) > (c)$, $(y) > (d)$, and consequently, $(xy) > (cy) > (cd)$. Because $cd \in Y$ by Proposition 2.1, it follows that $xy \notin P$, so P is prime in D. We note that if $x \in D \backslash P$, then x divides an element of Y, and because the multiplicative system Y is saturated, $x \in Y$. Therefore $D \backslash P \subseteq Y$. If $y \in Y$, then $(y) > (y^2) \supseteq P$, so $y \notin P$ and the equality $D \backslash P = Y$ holds.

It follows from (1) that the set of principal ideals of D/P is linearly ordered under inclusion. Consequently, D/P is a valuation domain.

To prove (3), take an element a/x of PD_P, where $a \in P$ and $x \in D \backslash P$. Then $(a) < (x)$, so $a = xy$ for some $y \in D$ that is necessarily in P. Thus $a/x = y \in P$ and $PD_P \subseteq P$.

(4): Since D/P and D_P are quasilocal, Proposition 2.2 shows that D is also quasilocal. To conclude that P is comparable, we need only show that $P \subseteq (x)$ for each $x \in D \backslash P$, and this is immediate from (1). Finally, the equality $\dim D = \dim(D/P) + \dim(D_P)$ follows from the fact that each prime ideal of D compares with P under inclusion.

Assume now that J is an integral domain satisfying (a) and (b). If $s \in Q$ and $t \in J \backslash Q$, then $s/t \in QJ_Q = Q$, and hence $s \in tQ \subseteq (t)$. Thus, to show t is comparable, we need only show that (t) compares with (u) for each $u \in J \backslash Q$. This follows because $Q \subseteq (t) \cap (u)$ and because the ideals $(t)/Q$ and $(u)/Q$ of the valuation domain J/Q are comparable. Therefore each element of $J \backslash Q$ is comparable. Let Q_0 be the intersection of the family $\{(b) \mid b$ is a nonzero comparable element of $J\}$. It follows from (1) that $J \backslash Q \subseteq J \backslash Q_0$, so $Q_0 \subseteq Q$. Moreover, (2) and (3) show that Q_0 satisfies (a) and (b). Thus, if Q is minimal with respect to satisfying (a) and (b), then $Q = Q_0$ and $J \backslash Q$ is the set of nonzero comparable elements of J. This completes the proof of Theorem 2.3. ∎

Let D, Y, and P be as in the statement of Theorem 2.3. In the literature, the domain D is called the *pullback* in D_P of the domain $D/P = D_P/PD_P$ [D], or the *composite* of D/P and D_P over PD_P [O], [MS]. Theorem 2.3 implies that the domains that admit nonzero nonunit comparable elements are precisely the pullbacks in R of nontrivial valuation domains on R/M, where (R, M) is any quasilocal domain (see also [GO, Prop. 5.1(f)]). Under the notation and hypothesis of Theorem 2.3, we note that if $P \neq (0)$, then neither D nor D_P is a valuation domain. For D this is clear, and for D_P the statement follows from the fact that the composite of two valuation domains is again a valuation domain [ZS, p.43], [N, p.35].

THEOREM 2.4. *In a quasilocal domain (D, M), the following conditions are equivalent for a nonzero element $x \in D$.*

(1) x is comparable.
(2) Each finitely generated ideal of D containing x is principal.
(3) Each strictly v-finite ideal of D containing x is principal.

Proof. (1)⇒(2): It suffices to show that $(x, d_1, \ldots, d_n)$ is principal for $d_1, \ldots, d_n \in D$. For $n = 1$, this follows since x is comparable. If $(x, d_1, \ldots, d_{n-1}) = (y)$ is principal, then y is comparable since y divides x, and by the case $n = 1$, $(x, d_1, \ldots, d_n) = (y, d_n)$ is also principal.

(2)⇒(3): Let A be a strictly v-finite ideal of D containing x. Thus there exists a finitely generated ideal $B \subseteq A$ such that $A_v = B_v$. Without loss of generality we can assume that $x \in B$, whence $B = (b)$ is principal by (2). Then $A_v = (b)_v = (b) \subseteq A \subseteq A_v$ and $A_v = (b)$ is principal.

(3)⇒(1): Pick an element $y \in D$. The ideal (x, y) is strictly v-finite, hence principal. Because D is quasilocal, it follows that $(x, y) = (x)$ or $(x, y) = (y)$ [G, Prop. 7.4]; consequently, $(y) \subseteq (x)$ or $(x) \subseteq (y)$ — that is, x is comparable. ∎

COROLLARY 2.5. *If the integral domain D contains a nonzero nonunit comparable element, then the maximal ideal of D is a t-ideal.*

Proof. Theorem 2.3 shows that D has a unique maximal ideal M. Let c be a nonzero nonunit comparable element of M. If $x_1, x_2, \ldots, x_n \in M$, then $(x_1, x_2, \ldots, x_n, c) = (m)$ is principal by Theorem 2.4, and $(x_1, \ldots, x_n)_v \subseteq (m) \subseteq M$. Therefore M is a t-ideal. ∎

We remark that to the equivalent conditions of Theorem 2.4, we could add: (4) M is a t-ideal, and each integral v-ideal of finite type containing x is principal.

COROLLARY 2.6. *A GCD-domain D contains a nonzero nonunit comparable element if and only if D is a valuation domain.*

Proof. We need only show that if D contains a nonzero nonunit comparable element c, then D is a valuation domain. Let M be the maximal ideal of D. Take nonzero elements x, y of M and let $(x, y)_v = (d)$. Write $x = x_1 d$, $y = y_1 d$. Then $(x_1, y_1)_v = D$. If $(x_1, y_1) \subseteq M$, then by Theorem 2.4, (x_1, y_1, c) is a proper principal ideal of D containing (x_1, y_1), and hence $(x_1, y_1)_v \subseteq (x_1, y_1, c) < D$. This contradiction shows that $(x_1, y_1) = D$, so x_1 or y_1 is a unit of D. We conclude that $(x, y) = (x)$ or $(x, y) = (y)$. Consequently, D is a valuation domain. ∎

3. t-INVERTIBILITY AND t-LOCAL COMPARABILITY

Suppose D is an integral domain and $d \in D - (0)$. We say that d is *t-locally comparable* if d is a comparable element of D_M for each maximal t-ideal M of D. Theorem 3.1 provides a connection between the concepts of t-invertibility and t-local comparability.

THEOREM 3.1. *Suppose D is an integral domain.*
(1) If A is a strictly v-finite ideal of D such that A contains a t-locally comparable element d, then A is t-invertible.
(2) Conversely, if $r \in D - (0)$ is such that each strictly v-finite ideal A of D containing r is t-invertible, then r is t-locally comparable.

Proof. (1): Suppose M is a maximal t-ideal of D containing A, and let B be a finitely generated ideal contained in A such that $A_v = B_v$. Without loss of generality we assume that $d \in B$. Then $((A_v)D_M)_v = (B_v D_M)_v = (BD_M)_v$ by [Z, Lemma 4]. Since d is a comparable element in D_M, Theorem 2.4 implies that $BD_M = aD_M$, where $a \in A$. Hence $aD_M \subseteq AD_M \subseteq (A_v D_M)_v = (BD_M)_v = (aD_M)_v = aD_M$. Therefore AD_M is principal for each maximal t-ideal containing A, and Corollary 1.6 of [MMZ] implies that A is t-invertible.

(2): Suppose M is a maximal t-ideal of D containing r. To show that r is a comparable element of D_M, it suffices, by Theorem 2.4, to show that each finitely generated ideal B of D_M containing r is principal. Now $B = AD_M$, where A is a finitely generated ideal of D containing r. By hypothesis, A is t-invertible, so $(AA^{-1})_t = D$. Since M is a maximal t-ideal, $AA^{-1} \not\subseteq M$. Therefore $AA^{-1}D_M = D_M$, so AD_M is invertible, and hence principal [G, Prop. 7.4]. ∎

Recall that an integral domain D is a *Prüfer v-multiplication domain* (PVMD) if the set of v-ideals of D of finite type form a group under v-multiplication. It is well known [Gr, Thm. 5] that D is a PVMD if and only if D_M is a valuation domain for each maximal t-ideal M of D. Theorem 3.2 provides a characterization of PVMD's in terms of t-local comparability.

THEOREM 3.2. *An integral domain D is a* PVMD *if and only if each nonzero prime ideal of D contains a t-locally comparable element.*

Proof. If D is a PVMD, then each nonzero element of D is t-locally comparable, so the condition holds. Conversely, suppose that each nonzero prime ideal of D meets S, the set of t-locally comparable elements of D. Proposition 2.1 implies that S is a saturated multiplicative system in D; hence each nonzero ideal of D meets S. Suppose A is a v-ideal of D of finite type. Then $A = B_v = B_t$ for some finitely generated ideal B. Because B is strictly v-finite, it is t-invertible by (1) of Theorem 3.1. Therefore $D = (BB^{-1})_t \subseteq (BB^{-1})_v = (AB^{-1})_v$, and A is v-invertible. Consequently, D is a PVMD. ∎

Recall that a nonzero element d of D is a *t-valuation element* if D_M is a valuation domain for each maximal t-ideal M containing d. Clearly a t-valuation element is t-locally comparable, and since each nonzero element of a PVMD is a t-valuation element, Theorem 3.2 has the following corollary.

COROLLARY 3.3. *An integral domain D is a* PVMD *if and only if each nonzero prime ideal of D contains a t-valuation element.*

Added in Proof. The referee has kindly pointed out to us that Proposition 2.2 is a direct consequence of results on glueing in the paper *Topologically defined classes of commutative rings*, Annali di Mat. Pura Applic. 123(1980), 331-355, by M. Fontana. ∎

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15
AF-Rings and Locally Jaffard Rings

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0 INTRODUCTION

AF-rings have been introduced by A.R. Wadsworth in [W] for computing the (Krull) dimension of the tensor product of A_1 and A_2, $\dim(A_1 \otimes_k A_2)$, where A_1 and A_2 are commutative algebras over a field k. Previously R.Y.Sharp proved in [S] that if K_1 and K_2 are extension fields of k, then

$$\dim(K_1 \otimes_k K_2) = \min\{\text{t.d.}(K_1:k)), \ \text{t.d.}(K_2:k)\}.$$

Since, for each prime ideal P of a k-algebra A, $\text{ht}(P) + \text{t.d.}(A/P:k) \le \text{t.d.}(A_P:k)$ (cf.[ZS, p.10] and [W, p.392]), for giving a generalization of Sharp's result, A.R. Wadsworth has studied the k-algebras A which satisfy the altitude formula over k, that is

$$\text{ht}(P) + \text{t.d.}(A/P:k) = \text{t.d.}(A_P:k)$$

for each prime ideal P of A and called these algebras AF-rings. In [W, Theorem 3.8.] it is shown that if D_1 and D_2 are AF-domains, then

$$\dim(D_1 \otimes_k D_2) = \min\{\text{t.d.}(D_1:k) + \dim(D_2), \ \dim(D_1) + \text{t.d.}(D_2:k)\}.$$

On the other hand in [ABDFK] Jaffard domains have been introduced and studied. We recall from [ABDFK, Definition 0.2.] that a finite-dimensional domain D is a Jaffard domain if $\dim(D[X_1, \dots, X_n]) = n + \dim(D)$ for each nonnegative integer n. The class of Jaffard domains is not stable under localizations and also in [ABDFK, Definition 1.4.] a domain D is defined to be a locally Jaffard domain if D_P is a Jaffard domain for each prime ideal P of D.