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t-linked extensions, the t-class group, and Nagata's theorem

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Abstract

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Let A be a subring of the integral domain B. Then B is said to be t-linked over A if for each finitely generated ideal I of A with $I^{-1} = A$, we have $(IB)^{-1} = B$. If A and B are Krull domains, this condition is equivalent to PDE. We show that if B is t-linked over A, then the map $I \to (IB)_t$ gives a homomorphism from the group of t-invertible t-ideals of A to the group of t-invertible t-ideals of B and hence a homomorphism $\mathrm{Cl}_t(A) \to \mathrm{Cl}_t(B)$ of the t-class groups. Conditions are given for these maps to be surjective which extend Nagata's Theorem for Krull domains to a much larger class of domains including, e.g., Noetherian domains each of whose grade-one prime ideals has height one.

Introduction

Let A be a Krull domain. Then the set of divisorial ideals of A is a (free abelian) group $D(A)$ under the v-product. If $A \subseteq B$ is an extension of Krull domains which satisfies PDE ($\mathrm{ht}(Q \cap A) \leq 1$ for each height-one prime ideal Q in B), then there is a natural homomorphism $D(A) \to D(B)$ which induces a

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homomorphism $\mathrm{Cl}(A) \to \mathrm{Cl}(B)$ of the divisor class groups. Examples of such extensions satisfying PDE are flat extensions, integral extensions, and subintersections. While in general the induced homomorphism $\mathrm{Cl}(A) \to \mathrm{Cl}(B)$ is neither injective nor surjective, Nagata's Theorem states that if $B = \bigcap_{P \in T} A_P$ is a subintersection of A, then the map $\mathrm{Cl}(A) \to \mathrm{Cl}(B)$ is surjective and the kernel is generated by the classes of the height-one prime ideals of A which are not in T.

Because the class group is defined only for completely integrally closed domains, its use has been limited mostly to results about Krull domains. The purpose of this paper is to utilize the recently introduced concepts of t-class group and t-linked extensions to extend these results as much as possible to general integral domains. It is hoped that the techniques introduced in this process will also deepen our understanding of Krull domains and their extensions which satisfy PDE.

In Section 1 we briefly describe the terms we shall use. The second section is then concerned with t-linked extensions. An integral domain B is said to be *t-linked* over a subring A (or the extension $A \subseteq B$ is t-linked) if for each finitely generated ideal I of A with $I^{-1} = A$, we have $(IB)^{-1} = B$. Several conditions equivalent to B being t-linked over A are given in Proposition 2.1. For Krull domains $A \subseteq B$, the notions of PDE and t-linkedness coincide. What is critical for the rest of the paper is the observation (Theorem 2.2) that if $A \subseteq B$ is t-linked, then the map $\theta : \mathrm{TI}(A) \to \mathrm{TI}(B)$ from the group of t-invertible t-ideals of A to that of B, given by $\theta(I) = (IB)_t$, is a homomorphism which induces a homomorphism $\bar{\theta} : \mathrm{Cl}_t(A) \to \mathrm{Cl}_t(B)$ of the t-class groups.

In the third section, we investigate three different natural maps $D(A) \to D(B)$, where $A \subseteq B$ are Krull domains. We show in Theorem 3.1 that a Noetherian Krull domain A is locally factorial if and only if the map $\theta : D(A) \to D(B)$, given by $\theta(I) = (IB)_v$, is a homomorphism for each Krull domain $B \supseteq A$.

In the final section we give three results related to Nagata's Theorem. We show (Theorem 4.5) that if A is a domain in which every prime t-ideal has height one (which is the case for Krull domains), then the homomorphisms $\mathrm{TI}(A) \to \mathrm{TI}(B)$ and $\mathrm{Cl}_t(A) \to \mathrm{Cl}_t(B)$ are surjective whenever $B = A_S$, S a multiplicatively closed subset of A. We also prove the surjectivity of these maps when A is a PVMD and B is a subintersection. The main result of this section, Theorem 4.8, is an analog of Nagata's Theorem for weakly Krull domains. Here A is said to be *weakly Krull* if $A = \bigcap_{P \in X^{(1)}(A)} A_P$ is a finite character intersection of localizations at height-one primes. We show that if $B = \bigcap_{Q \in T} A_Q$ $(T \subseteq X^{(1)}(A))$ is a subintersection of A, then the natural homomorphism $\mathrm{Cl}_t(A) \to \mathrm{Cl}_t(B)$ is surjective with kernel generated by the classes of the t-invertible t-ideals primary to primes in $X^{(1)}(A) \setminus T$. We also show that, unlike Krull domains, weakly Krull domains do not in general behave well under polynomial extensions but that a domain A which is simultaneously a weakly Krull domain and a Prüfer v-multiplication domain (a so-called generalized Krull domain) does have the property that $A[X]$ is again a weakly Krull Prüfer v-multiplication domain with t-class group equal to that of A (as is the case for Krull domains). Included also is a characterization of those weakly Krull domains A for which $A[X]$ is again weakly Krull.

1. Preliminaries

Let A be an integral domain with quotient field K. Recall that for a nonzero fractional ideal I of A, $I_v = (I^{-1})^{-1} = [A : [A : I]] = \bigcap \{xA \mid xA \supseteq I, x \in K\}$. An ideal I is said to be a *v-ideal*, *divisorial*, or *reflexive* if $I = I_v$. (Here, of course, by 'ideal' we mean 'fractional ideal'. For the rest of the paper, the context will make clear which meaning should be assigned to the word 'ideal'.) The set $D(A)$ of v-ideals is a monoid over the *v-product* $I * J = (IJ)_v$. Of course, $D(A)$ is a group if and only if A is completely integrally closed. For properties of the v-operation, the reader is referred to [19, Section 34].

However, we will be mostly interested in the *t-operation* $I \to I_t$, where $I_t = \bigcup \{J_v \mid J$ is a nonzero finitely generated subideal of $I\}$. (For properties of the t-operation, the reader may consult [2], [5], or [22].) An ideal I is called a *t-ideal* if $I = I_t$. A t-ideal (respectively, v-ideal) I has *finite type* if $I = (a_1, \ldots, a_n)_t$ (respectively, $I = (a_1, \ldots, a_n)_v$) for some finite subset $\{a_1, \ldots, a_n\}$ of I. While the set of v-ideals may be a proper subset of the set of t-ideals, the sets of finite type t-ideals and finite type v-ideals coincide. An ideal I is said to be *t-invertible* if there is an ideal J with $(IJ)_t = A$. If I is t-invertible, we may take $J = I^{-1}$. A t-invertible t-ideal necessarily has finite type; in fact, an ideal is t-invertible if and only if I_t has finite type and IA_P is principal for each maximal t-ideal P of A [23, Proposition 2.6]. The set $\mathrm{TI}(A)$ of t-invertible t-ideals of A is a group under the t-product $I * J = (IJ)_t$, and the set $P(A)$ of nonzero principal fractional ideals of A under multiplication is a subgroup of $\mathrm{TI}(A)$. The quotient group $\mathrm{Cl}_t(A) = \mathrm{TI}(A)/P(A)$ is called the *t-class group* of A; unlike the divisor class group, the t-class group is defined for arbitrary integral domains. When A is a Krull domain, the t- and the v-operations coincide, $D(A) = \mathrm{TI}(A)$, and $\mathrm{Cl}_t(A) = \mathrm{Cl}(A)$, the usual divisor class group of A. For properties of the t-class group, the reader is referred to [2, 3, 5, 8, 10, 11].

Our general reference for Krull domains and the divisor class group is [16]. For an integral domain A, $X^{(1)}(A)$ will denote the set of height-one prime ideals of A. Recall that for a Krull domain A, $D(A)$ is a free abelian group. An extension $A \subseteq B$ of Krull domains is said to satisfy *PDE* (*pas d'éclatement*) if for each $Q \in X^{(1)}(B)$, we have $\mathrm{ht}(Q \cap A) \leq 1$. The abbreviation *NBU* (for *no blowing up*) is sometimes used instead of PDE. Since in a Krull domain, a prime ideal is divisorial if and only if it has height one, the PDE condition can be restated in the following way: if Q is a prime t-ideal of B with $Q \cap A \neq 0$, then $(Q \cap A)_t \neq A$.

Recall that a *subintersection* of a domain A is an overring B of the form $B = \bigcap_{P \in T} A_P$ for some $T \subseteq \mathrm{Spec}(A)$. By [12, Proposition 4], T can always be restricted to a subset of the set of associated primes of A. In particular, T can be assumed to be a subset of the set of t-prime ideals of A, and, if A happens to be a Krull domain, we can further assume that $T \subseteq X^{(1)}(A)$. The statements of many of our results place such restrictions on T, but all subintersections considered are in fact arbitrary.

2. t-linked extensions

Let A be a subring of the integral domain B. Following [14], we say that B is *t-linked over* A if for each finitely generated fractional ideal I of A with $I^{-1} = A$, we have $(IB)^{-1} = B$. We list in Proposition 2.1 several conditions equivalent to B being t-linked over A. Note in particular that condition (3) shows that an extension of Krull domains $A \subseteq B$ is t-linked if and only if it satisfies PDE.

Proposition 2.1. *Let A be a subring of the integral domain B. Then the following statements are equivalent.*

(1) *B is t-linked over A.*
(2) *If I is a (finitely generated) ideal of A with $I_t = A$, then $(IB)_t = B$.*
(3) *If Q is a prime t-ideal of B with $Q \cap A \neq 0$, then $(Q \cap A)_t \neq A$.*
(4) *If Q is a maximal t-ideal of B with $Q \cap A \neq 0$, then $(Q \cap A)_t \neq A$.*
(5) *If I and J are t-invertible ideals of A with $I_t = J_t$, then $(IB)_t = (JB)_t$.*
(6) *If I is a t-invertible ideal of A, then $(IB)_t = (I_tB)_t$.*

Proof. The proof of [14, Proposition 2.1] shows that (1)–(3) are equivalent. (The parenthetical 'finitely generated' in statement (2) can be omitted, since for an ideal I with $I_t = A$ it follows that I_t has finite type.) That (3)$\Rightarrow$(4) is clear and the converse follows from the fact that a prime t-ideal is contained in a maximal t-ideal. It is also easy to see that (5)$\Rightarrow$(6)$\Rightarrow$(2). It remains only to show that (2)$\Rightarrow$(5). Assume (2). The equality $I_t = J_t$ yields $(IJ^{-1})_t = A$, which implies that $(IJ^{-1}B)_t = B$. Hence $(IB)_t = (IB(JJ^{-1}B)_t)_t = (JB(IJ^{-1}B)_t)_t = (JB)_t$. $\square$

Theorem 2.2. *Let $A \subseteq B$ be a pair of integral domains with B t-linked over A. Then the map $\theta : \mathrm{TI}(A) \to \mathrm{TI}(B)$, given by $\theta(I) = (IB)_t$, is a homomorphism. Furthermore, if xA is a principal fractional ideal of A, then $\theta(xA) = xB$; thus θ induces a homomorphism $\bar{\theta} : \mathrm{Cl}_t(A) \to \mathrm{Cl}_t(B)$, where $\bar{\theta}([I]) = [(IB)_t]$.*

Proof. We need only show that θ is a homomorphism. We have $\theta(I * J) = ((IJ)_tB)_t$. By Proposition 2.1(6), $((IJ)_tB)_t = (IJB)_t = ((IB)_t(JB)_t)_t = \theta(I) * \theta(J)$. $\square$

It should be noted that the homomorphism θ in Theorem 2.2 can exist without $A \subseteq B$ being t-linked. Indeed, let A be any domain with $\mathrm{Cl}_t(A) = 0$, and let B be any overring. Then for any t-invertible t-ideal I of A, we have that I is a principal ideal xA of A, whence $\theta(I) = xB \in \mathrm{TI}(B)$, and both the map θ and the induced map $\bar{\theta}$ are easily seen to be homomorphisms. For a specific example for which B is not t-linked over A, let $A = K[X, Y]$, where K is a field and X, Y are indeterminates over K, and let B be a valuation overring of A with maximal ideal M centered on (X, Y). Then M is a t-ideal of B with $M \cap A = (X, Y) \neq 0$, but $(X, Y)_t = A$. Thus $A \subseteq B$ is not t-linked. Examples of this type where both A and B are Krull domains will be discussed in the next section.

We next consider some examples of t-linked extensions. The following result, stated for the case of overrings in [14], gives several such examples.

Proposition 2.3. *Let A be an integral domain. Then:*

(1) *Any directed union of t-linked extensions of A is t-linked over A.*
(2) *Any intersection of t-linked extensions of A is t-linked over A.*
(3) *Any flat extension of A is t-linked over A.*
(4) *Any generalized transform of A is t-linked over A.*
(5) *The complete integral closure of A in its quotient field is t-linked over A.*

Proof. The proofs of (1)–(4) are given in [14, Proposition 2.2] for the case of overrings; these proofs easily extend to the general case (as is remarked in [14, Remark 2.5]). Statement (5) is proved in [14, Corollary 2.3]. □

It is well known that an extension $A \subseteq B$ of Krull domains satisfies PDE (is t-linked) if either (i) $A \subseteq B$ is flat, (ii) $A \subseteq B$ is integral, or (iii) B is a subintersection of A. Proposition 2.3 shows that any flat extension of integral domains is t-linked. Also, since an intersection of t-linked overrings is t-linked, $A \subseteq \bigcap A_{Q_\alpha}$ is t-linked for any collection $\{Q_\alpha\}$ of prime ideals of A; thus a domain is t-linked in any subintersection. The question of when an integral extension $A \subseteq B$ is t-linked is more delicate. If A is Noetherian and B is the integral closure of A in its quotient field, then $A \subseteq B$ is t-linked since in this case B is also the complete integral closure of A. (See [14, Corollary 2.14] for an extension of this to the case of quasicoherent A.) However, in general the integral closure of A in its quotient field need not be t-linked over A (see [15] for an example). We end this section by showing that a root-closed root extension is always t-linked.

Proposition 2.4. *Let $A \subseteq B$ be a root extension of integral domains (that is, for each $b \in B$, $b^n \in A$ for some positive integer n). Suppose further that B is root closed in its quotient field. Then the extension $A \subseteq B$ is t-linked.*

Proof. Let $I = (a_1, \ldots, a_k)$ be an ideal of A with $I^{-1} = A$. We shall show that $(IB)^{-1} = B$. Suppose that $xIB \subseteq B$ for some element x in the quotient field of B. There is a positive integer n such that $x^n(a_1^n, \ldots, a_k^n) \subseteq A$; it follows that x^n lies in the quotient field of A, and since $(a_1^n, \ldots, a_k^n)^{-1} = A$, we have $x^n \in A \subseteq B$. Since B is root closed, this gives $x \in B$, as desired. □

3. The case of Krull domains

In this section we consider maps from $\mathrm{TI}(A)$ to $\mathrm{TI}(B)$, where $A \subseteq B$ are Krull domains. For A a Krull domain, the t-operation and the v-operation are the same, so that $\mathrm{TI}(A) = D(A)$, the group of fractional divisorial ideals under the v-product $I * J = (IJ)_v$. Then $D(A)$ is a free abelian group on $X^{(1)}(A)$. Let $P(A)$ be the subgroup of $D(A)$ consisting of principal fractional ideals. Then the t-class group $\mathrm{Cl}_t(A) = \mathrm{TI}(A)/P(A) = D(A)/P(A) = \mathrm{Cl}(A)$, the usual divisor class group.

Suppose that $A \subseteq B$ is an extension of Krull domains. We wish to define a 'natural' homomorphism $\psi : D(A) \to D(B)$ with $\psi(P(A)) \subseteq P(B)$ so that ψ induces a homomorphism $\bar{\psi} : \mathrm{Cl}(A) \to \mathrm{Cl}(B)$. There are at least three natural ways to attempt to define such a map. We first consider the map given in the preceding section.

Define $\theta_1 : D(A) \to D(B)$ by $\theta_1(I) = (IB)_v$. Note that for x in the quotient field of A, $\theta_1(xA) = (xAB)_v = xB$, so that $\theta_1(P(A)) \subseteq P(B)$ and θ_1 takes the identity of $D(A)$ to the identity of $D(B)$. As was noted in the preceding section, θ_1 is a homomorphism if $A \subseteq B$ satisfies PDE (is t-linked). However, θ_1 need not in general be a homomorphism as is seen by Theorem 3.1. Note that the condition that θ_1 is a homomorphism translates to $((IJ)_v B)_v = (IJB)_v$ for all $I, J \in D(A)$. It follows that θ_1 is a homomorphism if and only if

$$\theta_1(P_1^{(n_1)} \cap \cdots \cap P_s^{(n_s)}) = ((P_1^{(n_1)} \cap \cdots \cap P_s^{(n_s)})B)_v = (P_1^{n_1} \cdots P_s^{n_s} B)_v$$

for $P_i \in X^{(1)}(A)$. Note that if A is locally factorial, then $D(A) = I(A)$, the group of invertible ideals of A with the usual ideal product (see [1, Theorem 1]). In this case, $\theta_1 : D(A) = I(A) \to I(B) \subseteq D(B)$ is a homomorphism. For Noetherian Krull domains, the converse is true.

Theorem 3.1. *For a Noetherian Krull domain A, the following statements are equivalent.*

(1) *A is locally factorial.*

(2) *For each Krull domain B containing A as a subring, the map $\theta_1 : D(A) \to D(B)$ is a homomorphism.*

(3) *For each overring B of A which is a DVR, the map $\theta_1 : D(A) \to D(B)$ is a homomorphism.*

Proof. We have already observed that (1) ⇒ (2), and it is clear that (2) ⇒ (3). To prove that (3) ⇒ (1), assume that A is not locally factorial. Then there is a noninvertible height-one prime P in A [1, Theorem 1]. Let x be a nonzero element of P, and write $xA = P_1^{(n_1)} \cap \cdots \cap P_s^{(n_s)}$, where $P_1 = P, P_2, \ldots, P_s$ are the height-one primes of A which contain xA. Note that we must have $xA = P_1^{(n_1)} \cap \cdots \cap P_s^{(n_s)} \supsetneqq P_1^{n_1} \cdots P_s^{n_s}$, lest $P = P_1$ be a factor of xA and hence be invertible. Thus $P_1^{n_1} \cdots P_s^{n_s} = C(xA) = C(P_1^{(n_1)} \cap \cdots \cap P_s^{(n_s)})$, where $C = P_1^{n_1} \cdots P_s^{n_s} : xA \neq A$. Let M be a maximal ideal containing C. By a result of Chevalley [13], there is a DVR-overring $(B, (\pi))$ of A with $\pi B \cap A = M$. Now if $\theta_1 : D(A) \to D(B)$ is a homomorphism, then

$$\begin{aligned} xB = \theta_1(xA) &= \theta_1(P_1^{(n_1)} \cap \cdots \cap P_s^{(n_s)}) \\ &= (P_1^{n_1} \cdots P_s^{n_s} B)_v = P_1^{n_1} \cdots P_s^{n_s} B = CxB = (CB)(xB)\,. \end{aligned}$$

Hence $CB = B$, and $B = CB \subseteq MB \subseteq \pi B$, a contradiction. □

We have seen that $\theta_1 : D(A) \to D(B)$ is a homomorphism if and only if $\theta_1(P_1^{(n_1)} \cap \cdots \cap P_s^{(n_s)}) = (P_1^{n_1} \cdots P_s^{n_s} B)_v$ for each finite subset $\{P_i\}$ of $X^{(1)}(A)$. We take this as our definition of the second map $\theta_2 : D(A) \to D(B)$. Since $D(A)$ is a free abelian group on $X^{(1)}(A)$, the set function $\theta_2 : X^{(1)}(A) \to D(B)$, given by $\theta_2(P) = (PB)_v$ extends to a unique homomorphism $\theta_2 : D(A) \to D(B)$. Note that since θ_2 is a homomorphism, $\theta_2(P_1^{(n_1)} \cap \cdots \cap P_s^{(n_s)}) = (P_1^{n_1} \cdots P_s^{n_s} B)_v$. Thus we see that θ_1 is a homomorphism if and only if $\theta_1 = \theta_2$. Hence if $A \subseteq B$ satisfies PDE, then $\theta_1 = \theta_2$.

Note that if $\theta_1 = \theta_2$, then $\theta_2(xA) = \theta_1(xA) = xB$ for each x in the quotient field of A, and so $\theta_2(P(A)) \subseteq P(B)$. However, we may have $\theta_2(xA) = xB$ for all nonzero elements of the quotient field of A without $A \subseteq B$ satisfying PDE. For example, take $A = K[X, Y]$, where K is a field and X, Y are indeterminates, and let B be a DVR centered on (X, Y). Then $\theta_1 = \theta_2$, since A is factorial, but certainly $A \subseteq B$ does not satisfy PDE. Also, note that we can have $\theta_2(P(A)) \subseteq P(B)$ without having $\theta_2(xA) = xB$ or $\theta_1 = \theta_2$. For example, if B is factorial, then $P(B) = D(B)$, and thus we certainly have $\theta_2(P(A)) \subseteq P(B)$. However, as the proof of Theorem 3.1 shows, even when B is a DVR, we may have $\theta_2(xA) \neq xB$ and $\theta_1 \neq \theta_2$.

The third (and customary) way to define a function $D(A) \to D(B)$ was given by Samuel [24] and Fossum [16]. We define this map in ideal-theoretic terms. As in the definition of θ_2, we define a function $\theta_3 : X^{(1)}(A) \to D(B)$ which then extends to a unique homomorphism $\theta_3 : D(A) \to D(B)$. Suppose that $P \in X^{(1)}(A)$. If $(PB)_v = B$, then define $\theta_3(P) = B$. If $(PB)_v \neq B$, then $(PB)_v = Q_1^{(n_1)} \cap \cdots \cap Q_r^{(n_r)} \cap Q_{r+1}^{(n_{r+1})} \cap \cdots \cap Q_s^{(n_s)}$, where each $Q_i \in X^{(1)}(B)$ and where $Q_i \cap A \supsetneqq P$ for $i = 1, \ldots, r$ and $Q_i \cap A = P$ for $i = r+1, \ldots, s$. Define $\theta_3(P) = Q_{r+1}^{(n_{r+1})} \cap \cdots \cap Q_s^{(n_s)}$ (and $\theta_3(P) = B$ if $r = s$).

Note that $\theta_2 = \theta_3$ if and only if $A \subseteq B$ satisfies PDE. Hence if $\theta_1 = \theta_3$, then θ_1 is a homomorphism, and so $\theta_1 = \theta_2 = \theta_3$ and $A \subseteq B$ satisfies PDE. Thus if $A \subseteq B$ satisfies PDE, then $\theta_3(xA) = xB$ for each x in the quotient field of A, so that

$\theta_3(P(A)) \subseteq P(B)$. However, the map θ_3 may satisfy $\theta_3(P(A)) \subseteq P(B)$ without having $\theta_2 = \theta_3$. For example, in the example discussed above, where $A = K[X, Y]$ and B is a DVR centered on (X, Y), θ_3 maps every element of $D(A)$ to B while θ_2 maps no element of $X^{(1)}(A)$ contained in (X, Y) to B. Thus, contrary to [16, Theorem 6.2], $\theta_3(P(A)) \subseteq P(B)$ does not imply that $A \subseteq B$ satisfies PDE. (Fossum has misquoted [24, Theorem 6.1], which, stated in ideal-theoretic terms, says that $A \subseteq B$ satisfies PDE if and only if $\theta_3(xA) = xB$ for all x in the quotient field of A.) Our next theorem summarizes the relationship between the maps θ_1, θ_2, and θ_3 and $A \subseteq B$ satisfying PDE. The relationship between θ_1 and θ_3 has also been considered by D.F. Anderson [7, Section 5]. However, he has communicated to us that his example with (his) ψ not a homomorphism is incorrect. He has suggested the following example of a pair of Krull domains not satisfying PDE but for which $\theta_3(P(A)) \subseteq P(B)$. Take $A = K[X, XY]$ and $B = K[X, Y]$. Then $XB \cap A = (X, XY)$ has height two in A but $\theta_3(P(A)) \subseteq P(B)$ since B is factorial.

Theorem 3.2. *For a pair $A \subseteq B$ of Krull domains, the following conditions are equivalent.*

(1) *$A \subseteq B$ satisfies PDE; i.e., $Q \in X^{(1)}(B)$ implies* $\mathrm{ht}(Q \cap A) \leq 1$.

(2) *For an ideal I of A, $I_v = A$ implies $(IB)_v = B$; i.e., B is t-linked over A.*

(3) *For (fractional) ideals I and J of A, $I_v = J_v$ implies $(IB)_v = (JB)_v$.*

(4) *For a (fractional) ideal I of A, $(IB)_v = (I_vB)_v$.*

(5) $\theta_1 = \theta_3$.

(6) $\theta_2 = \theta_3$.

(7) $\theta_1 = \theta_2 = \theta_3$.

(8) $\theta_3(xA) = xB$ *for each nonzero element $x \in A$.*

Proof. The equivalence of statements (1)–(4) follows from Proposition 2.1. We have already observed that (1)$\Leftrightarrow$(6). Certainly (5)$\Leftrightarrow$(7) since $\theta_1 = \theta_2$ if and only if θ_1 is a homomorphism. That (7)$\Rightarrow$(6) is clear; and if (6) holds, then from (6)$\Leftrightarrow$(1) we get that $\theta_1 = \theta_2$, and so (7) holds. Certainly, (5)$\Rightarrow$(8). Assume that (8) holds. Let $Q \in X^{(1)}(B)$ with $Q \cap A \neq 0$. Let $0 \neq x \in Q \cap A$. Since $\theta_3(xA) = xB$, we must have that Q contracts to a prime minimal over xA, and so $\mathrm{ht}(Q \cap A) = 1$. Thus (8)$\Rightarrow$(1), and the proof is complete. □

4. Extensions of Nagata's Theorem and weakly Krull domains

The purpose of this section is to give several extensions of Nagata's Theorem and to study polynomial extensions of weakly Krull domains. We begin by stating Nagata's Theorem and two of its important corollaries. For proofs and the history of these results, the reader is referred to [16, Section 7].

Theorem 4.1. (Nagata's Theorem) *Let A be a Krull domain and suppose that $B = \bigcap_{P \in T} A_P$ is a subintersection of A (where it may be assumed that $T \subseteq$*

$X^{(1)}(A)$). *Then the homomorphism* $\theta : \mathrm{Cl}(A) \to \mathrm{Cl}(B)$, *given by* $\theta([I]) = [(IB)_v]$ *is surjective and* ker θ *is generated by the classes of those height-one primes of* A *not in* T. □

Corollary 4.2. *Let* S *be a multiplicatively closed subset of the Krull domain* A. *Then the homomorphism* $\theta : \mathrm{Cl}(A) \to \mathrm{Cl}(A_S)$ *is surjective and* ker θ *is generated by the classes of height-one primes of* A *that meet* S. □

Corollary 4.3. *Let* S *be a multiplicatively closed subset of the Krull domain* A. *If* S *is generated by principal primes, then the homomorphism* $\theta : \mathrm{Cl}(A) \to \mathrm{Cl}(A_S)$ *is an isomorphism.* □

Now let A be an integral domain, not necessarily a Krull domain, and let $B = \bigcap_{P \in T} A_P$ be a subintersection of A. Then $A \subseteq B$ is a t-linked extension, and so we have by Theorem 2.2 a homomorphism $\theta : \mathrm{TI}(A) \to \mathrm{TI}(B)$ given by $\theta(I) = (IB)_t$. Since for x in the quotient field of A, $\theta(xA) = xB$, θ induces a homomorphism $\bar{\theta} : \mathrm{Cl}_t(A) \to \mathrm{Cl}_t(B)$ with $\bar{\theta}([I]) = [(IB)_t]$. Two natural questions have received wide attention. The first simply asks whether $\bar{\theta}$ is surjective. The second question asks, in the case where $B = A_S$, S a multiplicatively closed subset of A generated by principal primes, whether $\bar{\theta}$ is an isomorphism? In [2, Theorem 2.3] it was shown that $\bar{\theta}$ is injective when S is generated by principal primes, but an example was given of a domain D and an element $f \in D$ for which $\mathrm{Cl}_t(D) \to \mathrm{Cl}_t(D_f)$ is not surjective. In [18] Gabelli and Roitman studied conditions under which $\bar{\theta}$ is surjective; they showed that $\bar{\theta}$ is surjective (and hence an isomorphism) when S is generated by (what is called in [3]) a splitting set of principal primes (that is, a set $\{p_\alpha\}$ of nonassociate principal primes with $\bigcap_{n=1}^{\infty} p_\alpha^n A = 0$ for each α and $\bigcap_{k=1}^{\infty} p_{\alpha_k} A = 0$ for each countably infinite subset $\{p_{\alpha_k}\}$ of $\{p_\alpha\}$). They also gave an example in which S was generated by principal primes, but for which $\bar{\theta}$ was not surjective. Their result was proved independently in [3] using entirely different techniques. The reader is referred to [3] for further discussion of these questions. We remark that D.F. Anderson and A. Ryckaert [9] have shown that for any two abelian groups G and H, there is an integral domain A and a multiplicatively closed subset S of A with $\mathrm{Cl}_t(A) = G$ and $\mathrm{Cl}_t(A_S) = H$.

Our first result extends Theorem 4.1 to Prüfer v-multiplication domains (PVMD's). Recall that a PVMD is an integral domain in which every finitely generated ideal is t-invertible. Equivalently, a domain A is a PVMD if A_P is a valuation domain for each prime t-ideal P of A. In particular, Krull domains are PVMD's.

Theorem 4.4. *Let* A *be a PVMD, and let* $B = \bigcap_{P \in T} A_P$ *be a subintersection of* A, *where* T *is a subset of the set of t-primes of* A. *Then the map* $\theta : \mathrm{TI}(A) \to \mathrm{TI}(B)$, *given by* $\theta(I) = (IB)_t$ *is a surjective homomorphism. The induced map* $\bar{\theta} : \mathrm{Cl}_t(A) \to \mathrm{Cl}_t(B)$ *is also a surjective homomorphism.*

Proof. It suffices to show that θ is surjective. Let $J=(x_1,\ldots,x_n)_v$ be a t-invertible t-ideal of B; we may assume that J is an integral ideal of B. Set $I=A:x_1\cap\cdots\cap A:x_n$. (Here, $A:x=\{a\in A \mid ax\in A\}$.) Then I is a finite-type t-ideal of A. We claim that $(IB)_t=B$. To verify this, suppose that u is an element of the quotient field of A for which $uI\subseteq B$. Then for each $P\in T$, we have $uI\subseteq A_P$. However, $I\not\subseteq P$, since each $x_i\in A_P$. Hence $u\in\bigcap A_P=B$. Thus $(IB)_v=B$. Now $I=C_v$ for some finitely generated ideal C of A, and by Proposition 2.1, (1)$\Rightarrow$(6), $(C_vB)_v=(CB)_v$. Thus $(IB)_t=B$ (since $(CB)_v=B$, and CB is a finitely generated subideal of IB). Put $I_1=Ix_1+\cdots+Ix_n$. Since $(I_1)_t$ has finite type and A is a PVMD, $(I_1)_t$ is t-invertible. We shall show that $\theta((I_1)_t)=J$. Note that $(I_1)_t=(C_vx_1+\cdots+C_vx_n)_v=(Cx_1+\cdots+Cx_n)_v=(Cx_1+\cdots+Cx_n)_t$. Hence, again applying Proposition 2.1, we have $((I_1)_tB)_t=((Cx_1+\cdots+Cx_n)B)_t\subseteq(IJ)_t=((IB)_tJ)_t=J$. On the other hand,

$$\begin{aligned}((I_1)_tB)_t&=((Ix_1+\cdots+Ix_n)_tB)_t\supseteq((Ix_1+\cdots+Ix_n)B)_t\\&=(IB(x_1,\ldots,x_n))_t=((IB)_t(x_1,\ldots,x_n))_t=J\,,\end{aligned}$$

and the proof is complete. □

One of the important properties of Krull domains is that their divisorial primes have height one. The t-ideal analog is the requirement that each prime t-ideal (equivalently, each maximal t-ideal) have height one. A domain A with this property is said to have *t-dimension* 1 (written t-dim$(A)=1$). The second main result of this section is that if t-dim$(A)=1$, then $\bar{\theta}:\mathrm{Cl}_t(A)\to\mathrm{Cl}_t(A_S)$ is indeed surjective. This result is a generalization of (and our proof borrows heavily from that of) [18, Theorem 1.18].

Theorem 4.5. *Let A be an integral domain with* t-dim$(A)=1$ *(so that every maximal t-ideal of A has height one), and let S be a multiplicatively closed subset of A. Then the map $\theta:\mathrm{TI}(A)\to\mathrm{TI}(A_S)$, given by $\theta(I)=(IA_S)_t$, is a surjective homomorphism. Hence the induced map $\bar{\theta}:\mathrm{Cl}_t(A)\to\mathrm{Cl}_t(A_S)$ is also a surjective homomorphism.*

Proof. Again, it suffices to show that θ is surjective. Let J be a t-invertible t-ideal of A_S. Then $J=(IA_S)_t$ for some finitely generated ideal I of A. If $(II^{-1})_t\cap S=\emptyset$, then $(II^{-1})_t$ can be expanded to a prime ideal P maximal with respect to avoiding S and being a t-ideal. By hypothesis, ht$(P)=1$, whence PA_S is a height-one prime, and therefore a prime t-ideal, of A_S. However, $(JJ^{-1})_t=((IA_S)_t(IA_S)^{-1})_t=(IA_SI^{-1}A_S)_t=(II^{-1}A_S)_t\subseteq PA_S$, a contradiction. Hence there is an element $s\in(II^{-1})_t\cap S$. Put $I_1=IA_s\cap A$. (Here $A_s=\{\frac{a}{s^n}\mid a\in A$ and n is a nonnegative integer$\}$.) Then, since s is not a zero-divisor mod I_1, the ideal (I_1,s) lies in no height-one prime of A. Thus, since t-dim$(A)=1$, $(I_1,s)_t=A$. Let $I_2=I+I_1'$, where I_1' is a finitely generated ideal of A with $I_1'\subseteq I_1$ and $(I_1',s)_t=A$.

Clearly, $I_2A_S = IA_S$. Since I_2 is finitely generated, we have that $((I_2)_tA_S)_t = (I_2A_S)_t = (IA_S)_t = J$; we shall complete the proof by showing that I_2 is t-invertible. Since $s \in (II^{-1})_t$; there are elements $a_1, \ldots, a_n \in I$ and $u_1, \ldots, u_n \in I^{-1}$ with $s \in ((a_1, \ldots, a_n)(u_1, \ldots, u_n))_v$. Let $x \in I_1' \subseteq I_1 = IA_S \cap A$. Then for some positive integer k, $s^kx \in I$ and $s^kxu_i \in A$ for each i. Thus $s^ku_iI_1' \subseteq A$ for sufficiently large k. Since $s^ku_iI \subseteq A$ also, we have $s^ku_i \in I_2^{-1}$. Thus

$$\begin{aligned} s^{k+1} &\in s^k((a_1, \ldots, a_n)(u_1, \ldots, u_n))_v \\ &= ((a_1, \ldots, a_n)(s^ku_1, \ldots, s^ku_n))_v \subseteq (I_2I_2^{-1})_t . \end{aligned}$$

Hence $(I_2I_2^{-1})_t \supseteq (I_2, s^{k+1})_t \supseteq (I_1', s^{k+1})_t = A$, and I_2 is t-invertible. □

Remark. We do not know whether the analogue of Theorem 4.5 can be proved for subintersections.

Example 4.6. We give an example of a Noetherian domain A for which the map $\mathrm{Cl}_t(A) \to \mathrm{Cl}_t(B)$ is surjective for each subintersection B of A but for which the t-dimension of A is greater than 1. Let R be a Noetherian UFD with exactly n maximal ideals $N_1, \ldots, N_n$ such that each R/N_i is isomorphic to a fixed field K. For each i let ε_i be a surjective homomorphism from R to K with kernel N_i. Set $A = \{r \in R \mid \varepsilon_i(r) = \varepsilon_j(r) \text{ for all } i, j\}$. By [25, Theorem, p. 585], A is a local Noetherian domain with maximal ideal $M = N_1 \cap \cdots \cap N_n$ such that M is the conductor of A in R and such that R is the integral closure of A. Thus M is divisorial (and therefore a t-ideal) in A and $\mathrm{ht}(M) = \max\{\mathrm{ht}(N_i)\}$. Now we can arrange to have the N_i of equal height greater than one, so that we can assume that $\text{t-dim}(A) \geq 2$. For each prime P of A for which $P \neq M$, we have $A_P = R_P$, so that any subintersection B of A which properly contains A is actually a subintersection of R. Thus $\mathrm{Cl}_t(B) = 0$, and $\mathrm{Cl}_t(A) \to \mathrm{Cl}_t(B)$ is surjective, for each such B. □

An integral domain A is said to be *weakly Krull* if $A = \bigcap_{P \in X^{(1)}(A)} A_P$, where the intersection has finite character. Weakly Krull domains (although not called that there) were introduced in [5]. Weakly Krull domains share many properties with Krull domains. For example, in a weakly Krull domain A, $X^{(1)}(A)$ is the set of prime t-ideals, that is, t-dim $A = 1$ [5, Lemma 2.1]. It follows from [5, Theorem 3.1] that a domain A is weakly Krull if and only if every t-invertible t-ideal is a t-product of primary t-ideals. Moreover, if I is a t-invertible t-ideal of a weakly Krull domain A, then $I = Q_1 \cap \cdots \cap Q_n = (Q_1 \cdots Q_n)_t$, where $Q_i = IA_{P_i} \cap A$ with $\{P_1, \ldots, P_n\}$ the set of height-one primes containing I and each Q_i a P_i-primary t-invertible t-ideal.

Let A be weakly Krull. Then every subintersection B of A has the form $B = \bigcap_{Q \in T} A_Q$ for some subset T of $X^{(1)}(A)$. Note that by Proposition 2.3, B is t-linked over A. Thus if I is a t-invertible t-ideal of A, then IB is a t-invertible ideal of B and by Theorem 2.2 the map $\theta : \mathrm{TI}(A) \to \mathrm{TI}(B)$, given by $\theta(I) = (IB)_t$, is a homomorphism. We show that B is weakly Krull.

Proposition 4.7. *Let A be weakly Krull and let $B = \bigcap_{Q \in T} A_Q$ be a subintersection (where T is a nonempty subset of $X^{(1)}(A)$). Then B is also weakly Krull. Moreover, $X^{(1)}(B) = \{B \cap QB_Q \mid Q \in T\}$ and for $Q \in X^{(1)}(B)$, $B_Q = A_{Q \cap A}$.*

Proof. Let P be a prime t-ideal of B. Since $P \cap A \neq 0$, $(P \cap A)_t \neq A$ (because B is t-linked over A). Hence $\mathrm{ht}(P \cap A) = 1$, since A is weakly Krull. Now $A_{P \cap A} \subseteq B_P = (\bigcap_{Q \in T} A_Q)_P = \bigcap_{Q \in T} (A_Q)_{B \setminus P}$, since the intersection has finite character. Each $(A_Q)_{B \setminus P}$ is either A_Q or the quotient field of A. Thus $A_{P \cap A} \subseteq B_P \subseteq A_Q$ for some $Q \in T$. But then $Q = P \cap A$, $A_{P \cap A} = B_P$, and $\mathrm{ht}(P) = 1$. If $Q \in T$, then $Q' = B \cap QA_Q$ is a prime ideal of B and $A_Q \subseteq B_{Q'} \subseteq A_Q$, and so $B_{Q'} = A_Q$ and $\mathrm{ht}(Q') = 1$. $\square$

We are now prepared to prove the main result of this paper: the extension of Nagata's Theorem to weakly Krull domains.

Theorem 4.8. (Nagata's Theorem) *Let A be a weakly Krull domain and let $B = \bigcap_{P \in T} A_P$ be a subinteraction of A with $\emptyset \neq T \subseteq X^{(1)}(A)$. Then the homomorphism $\theta : \mathrm{TI}(A) \to \mathrm{TI}(B)$, given by $\theta(I) = (IB)_t$, is surjective. Hence the induced homomorphism $\bar{\theta} : \mathrm{Cl}_t(A) \to \mathrm{Cl}_t(B)$ is also surjective; moreover, its kernel is generated by the classes of t-invertible t-ideals primary to primes from $X^{(1)}(A) \setminus T$.*

Proof. We have already observed that θ is a homomorphism. To show that θ is surjective, it suffices to prove that if J is a t-invertible t-ideal of B, then $J \cap A$ is t-invertible in A and $J = ((J \cap A)B)_t$. Let $\mathcal{P}_1, \ldots, \mathcal{P}_n$ be the height-one primes of B which contain J. Then $J = Q_1 \cap \cdots \cap Q_n$, where $Q_i = JB_{\mathcal{P}_i} \cap B$. Note that for each i, Q_i is $\mathcal{P}_i$-primary and $JB_{\mathcal{P}_i}$ in principal (since J is t-invertible). Put $P_i = \mathcal{P}_i \cap A$. Then $J \cap A = (Q_1 \cap A) \cap \cdots \cap (Q_n \cap A)$, each $Q_i \cap A$ is P_i-primary, and $\{P_1, \ldots, P_n\}$ is the set of (height-one) primes of A minimal over $J \cap A$. Since $(J \cap A)A_{P_i} = JA_{P_i} = JB_{\mathcal{P}_i}$ is principal, $J \cap A$ is t-invertible [5, Lemma 2.2] and is in fact a t-ideal since each $Q_i \cap A$ is a t-ideal [5, Corollary 2.3]. Now $(J \cap A)B$ is t-invertible, and for $\mathcal{P} \in X^{(1)}(B)$, we have $((J \cap A)B)B_{\mathcal{P}} = (J \cap A)B_{\mathcal{P}} = (J \cap A)A_{\mathcal{P} \cap A} = JA_{\mathcal{P} \cap A} = JB_{\mathcal{P}}$. Thus, since both $(J \cap A)B$ and J are t-invertible, we have

$$((J \cap A)B)_t = \bigcap_{\mathcal{P} \in X^{(1)}(B)} (J \cap A)B_P = \bigcap_{\mathcal{P} \in X^{(1)}(B)} JB_{\mathcal{P}} = J_t = J .$$

Now suppose that Q is a t-invertible P_0-primary t-ideal, where $P_0 \in X^{(1)}(A) \setminus T$. Let $\mathcal{P} \in X^{(1)}(B)$, and set $P = \mathcal{P} \cap A$. Then $P \neq P_0$, and $(QB)B_{\mathcal{P}} = QA_P = A_P$, whence $(QB)_t = B$ and $[Q] \in \ker \bar{\theta}$. On the other hand, suppose that $[I] \in \ker \bar{\theta}$.

We may assume that I is an integral t-invertible t-ideal of A. Then $(IB)_t$ is principal; say, $(IB)_t = \frac{a}{b}B$ for some $a,b \in A$. Then $(\frac{b}{a}IB)_t = B$, that is, $\frac{b}{a}IA_P = A_P$ for each $P \in T$. Hence $bIA_P = aA_P$ for each $P \in T$. Since A is weakly Krull, we may write

$$bI = \left(\prod_{i=1}^{n} (bIA_{P_i} \cap A)\right)_t \quad \text{and} \quad aA = \left(\prod_{j=1}^{m} (aA_{Q_j} \cap A)\right)_t .$$

Then $a^{-1}A = (\prod_{j=1}^{m} (aA_{Q_j} \cap A)^{-1})_t$. This gives

$$\tfrac{b}{a}I = \left(\prod_{i=1}^{n} (bIA_{P_i} \cap A)\right)_t \left(\prod_{j=1}^{m} (aA_{Q_j} \cap A)^{-1}\right)_t .$$

Note that if some $P_i \in T$, then P_i is one of the Q_j; similarly, if some $Q_k \in T$, then Q_k is one of the P_r. Moreover, the corresponding t-invertible primary factors of bI and aA are then equal. Thus for all $P \in T$, any t-invertible factor of bI primary to a prime in T must cancel out with the corresponding t-invertible primary factor of aA. Deleting these factors we see that $\frac{b}{a}I = (N_1 \cdots N_r N_{r+1}^{-1} \cdots N_s^{-1})_t$, where each N_i is a t-invertible t-ideal primary to a prime in $X^{(1)}(A) \backslash T$. □

It is well known that a Krull domain A is a UFD if and only if $\mathrm{Cl}(A) = 0$. An easy consequence of this fact and Corollary 4.3 is that if A is a Krull domain and S a multiplicatively closed subset generated by principal primes for which A_S is a UFD, then A is a UFD. In [4] the concept of weak factoriality was introduced; a domain A is *weakly factorial* if each nonzero nonunit of A is a product of primary elements. In [6] it was shown that A is weakly factorial if and only if A is a weakly Krull domain for which $\mathrm{Cl}_t(A) = 0$. Although a deep treatment of weak factoriality would be out of place in this work, we offer the following analogue of Corollary 4.3. These ideas are treated more generally in [3].

Corollary 4.9. *Let A be a weakly Krull domain, and let S be a multiplicatively closed subset of A which is generated by principal primes. Then the homomorphism $\bar{\theta} : \mathrm{Cl}_t(A) \to \mathrm{Cl}_t(A_S)$ is an isomorphism. In particular, if A_S is weakly factorial, then so is A.*

Proof. As already mentioned, [2, Theorem 2.3] shows that $\bar{\theta}$ is injective. Hence the conclusion follows from Theorem 4.8. □

It is well known that for a Krull domain A, $A[X]$ is a Krull domain and the homomorphism $\bar{\theta} : \mathrm{Cl}(A) \to \mathrm{Cl}(A[X])$ is an isomorphism. Now for any integral domain A, the extension $A \subseteq A[X]$ is t-linked; so we have homomorphisms $\theta : \mathrm{Tl}(A) \to \mathrm{Tl}(A[X])$ and $\bar{\theta} : \mathrm{Cl}_t(A) \to \mathrm{Cl}_t(A[X])$. Unlike the case where A is a Krull domain, however, $\bar{\theta}$ need not be an isomorphism. In fact, Gabelli [17] has shown that $\bar{\theta}$ is an isomorphism if and only if A is integrally closed. This leads to the question of whether A weakly Krull implies $A[X]$ weakly Krull. The answer is no; we give the exact relationship in Proposition 4.11. First, recall a concept introduced in [21]. A domain A is said to be a *UMT-domain* if every upper to zero (a nonzero prime of $A[X]$ which contracts to zero in A) Q of $A[X]$ is a maximal t-ideal (equivalently, is t-invertible).

Lemma 4.10. *Let A be an integral domain. Then* t-dim$(A[X]) = 1$ *if and only if* t-dim$(A) = 1$ *and A is a UMT-domain.*

Proof. First, assume that t-dim$(A[X]) = 1$, and let P be a prime t-ideal of A. Then $PA[X]$ is a prime t-ideal of $A[X]$, and ht$(PA[X]) = 1$. Thus ht$(P) = 1$. Hence t-dim$(A) = 1$. If Q is an upper to zero in $A[X]$, then Q is a maximal t-ideal since t-dim$(A[X]) = 1$. Thus A is a UMT-domain. To prove the converse, suppose that N is a maximal t-ideal of $A[X]$ with $N \cap A \neq 0$. Then $N = (N \cap A)A[X]$ by [21, Proposition 1.1]. Since $N \cap A$ is a t-ideal, it has height one, so that ht$(N) \leq 2$. However, if Q is an upper to zero, then (since A is a UMT-domain) Q is a maximal t-ideal, so that $Q \not\subseteq N$. It follows that ht$(N) = 1$, as desired. □

Proposition 4.11. *Let A be an integral domain. Then A[X] is weakly Krull if and only if A is a weakly Krull UMT-domain.*

Proof. Assume that $A[X]$ is weakly Krull. Then $X^{(1)}(A[X]) = \mathscr{E} \cup \mathscr{U}$, where $\mathscr{E}$ is the set of extensions to $A[X]$ of height-one primes of A and $\mathscr{U}$ is the set of uppers to zero. Let u be an element of K, the quotient field of A, and assume that $u \in \bigcap_{P \in X^{(1)}(A)} A_P$. Then it is clear that

$$u \in \bigcap_{Q \in \mathscr{E}} A[X]_Q \cap \Big(\bigcap_{Q \in \mathscr{U}} A[X]_Q \Big) = \bigcap_{Q \in X^{(1)}(A[X])} A[X]_Q = A[X] .$$

Hence $u \in A[X] \cap K = A$. Since the intersection clearly has finite character, A is weakly Krull.

For the converse, note that t-dim$(A[X]) = 1$ by Lemma 4.10. Using the notation of the preceding paragraph, a nonzero element f of $A[X]$ lies in only finitely many elements of $\mathscr{E}$ since A is weakly Krull, and it is clear that f lies in only finitely many elements of $\mathscr{U}$. Thus f lies in only finitely many primes in $X^{(1)}(A[X])$, and so $A[X]$ is weakly Krull. □

For an example of a weakly Krull domain A for which $A[X]$ is not weakly Krull, let (A, M) be a one-dimensional quasilocal domain whose integral closure is not a Prüfer domain. Then A is trivially weakly Krull. However, dim$(A[X]) = 3$ [19, Proposition 30.14], and it follows that $A[X]$ contains an upper to zero Q such that $Q \subsetneq MA[X]$. Thus $A[X]$ is not a UMT-domain. However, since a Noetherian domain is a UMT-domain if and only if it has t-dimension equal to one [21, Theorem 3.7] (if and only if every grade-one prime has height one), the situation for Noetherian domains is just like that for Krull domains:

Corollary 4.12. *Let A be a Noetherian domain. Then A is weakly Krull if and only $A[X]$ is weakly Krull.* □

As mentioned above, if A is a Krull domain, then $A[X]$ is a Krull domain with the same class group (up to isomorphism). We have seen that there is no analogue of this result for weakly Krull domains. The following result characterizes those weakly Krull domains for which such an analogue exists. Recall that an integral domain A is a *generalized Krull domain* if A is a finite character intersection of rank-one essential valuation overrings [19, Definition 43.1].

Corollary 4.13. *For a domain A the following conditions are equivalent.*
(1) *A is a generalized Krull domain.*
(2) *A is a weakly Krull PVMD.*
(3) *$A[X]$ is weakly Krull and* $\mathrm{Cl}_t(A) = \mathrm{Cl}_t(A[X])$.

Proof. If A is a generalized Krull domain, then A is weakly Krull by [19, Corollary 43.9], and A is a PVMD by [20, Theorem 7]. It is clear that a weakly Krull PVMD is a generalized Krull domain. Hence (1) and (2) are equivalent. According to [21, Proposition 3.2], A is a PVMD if and only if A is an integrally closed UMT-domain. Since A is integrally closed if and only if $\mathrm{Cl}_t(A) = \mathrm{Cl}_t(A[X])$ [17], the equivalence of (2) and (3) follows easily. □

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