# $t$-SCHREIER DOMAINS

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ABSTRACT. We study, under the name $t$-Schreier, the class of those integral domains whose group of $t$-invertible $t$-ideals satisfies the Riesz interpolation property. The so called PVMDs and Prüfer domains are special cases of $t$-Schreier domains. We show that while a number of results known for Prüfer domains and PVMDs hold for these domains, the $t$-Schreier domains have a remarkable capability of unifying various results in that the results proved for t-Schreier domains can also be translated to results on pre-Schreier domains and hence on GCD and Bezout domains.

## 1. INTRODUCTION.

The aim of this paper is to study integral domains $D$ whose groups $Inv_t(D)$ of $t$-invertible $t$-ideals under $t$-multiplication are Riesz groups. We give the name $t$-Schreier to these domains. Special cases of these domains are the Prüfer domains and the so-called Prüfer $v$-multiplication domains (PVMDs). A working introduction to the not so well known notions will be provided below, before a more technical description of our plan. For now, we characterize these domains, indicate their links with the notions currently in literature, study their behavior under localizations and polynomial extensions, and study when they are of finite $t$-character. Recall that a domain $D$ is of finite $t$-character if every nonunit of $D$ belongs to at most a finite number of maximal $t$-ideals.

Let $G$ be a directed partially ordered abelian group. $G$ is called a *Riesz group* if whenever $x \leq a_1 + a_2$ with $x, a_1, a_2 \geq 0$ elements of $G$, there exist $x_1, x_2 \in G$ such that $x = x_1 + x_2$ and $0 \leq x_i \leq a_i$, $i = 1, 2$. These groups were initially studied by various mathematicians as directed groups $G$ that satisfy the Riesz interpolation property (RIP): given that $x_1, x_2, ..., x_m; y_1, y_2, ..., y_n \in G$ such that $x_i \leq y_j$ for all $i \in [1, m], j \in [1, n]$ there is $z \in G$ such that $x_i \leq z \leq y_j$ for all $(i, j) \in [1, m] \times [1, n]$. To our knowledge, these groups were studied for their own sake by Roger Teller [31], where the equivalence of our definition and RIP can be found and in [16], and that these papers appeared almost simultaneously. The next theorem [16, Theorem 2.2] gives several characterizations of this concept.

**Theorem 1.** *For a directed ordered abelian group $G$, the following assertions are equivalent:*

(*a*) *$G$ is a Riesz group.*

(*b*) *For all $a_i, b_j \in G$ with $a_i \leq b_j$, $i, j = 1, 2$, there exists $c \in G$ such that $a_i \leq c \leq b_j$, $i, j = 1, 2$.*

(*c*) *For all $m, n \geq 2$ and $a_i, b_j \in G$ with $a_i \leq b_j$, $i \in [1, m]$, $j \in [1, n]$, there exist $c \in G$ such that $a_i \leq c \leq b_j$, $(i, j) \in [1, m] \times [1, n]$.*

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Let $D$ be a domain. $D$ is called a *pre-Schreier domain* cf. [12] and [33] if the group of nonzero principal fractional ideals of $D$ (ordered by reverse inclusion) is a Riesz group. $D$ is called a *quasi-Schreier domain* cf. [13] if the group of nonzero invertible fractional ideals of $D$ (ordered by reverse inclusion) is a Riesz group. In this article we study domains $D$ whose groups of $t$-invertible $t$-ideals, under $t$-multiplication, ordered by reverse inclusion, are Riesz groups. Obviously, this new concept depends upon the notion of star operations, the $v$-operation and the $t$-operation. We assume this knowledge and refer the reader to sections 32 and 34 of [18] for a quick reference. For our purposes we give a working introduction here, to the notions involved.

Throughout, the letter $D$ denotes an integral domain with quotient field $K$ and $F(D)$ denotes the set of nonzero fractional ideals. A star operation $*$ on $D$ is a function $*\colon F(D) \to F(D)$ such that for all $A, B \in F(D)$ and for all $0 \neq x \in K$

(a) $(x)^* = (x)$ and $(xA)^* = xA^*$,

(b) $A \subseteq A^*$ and $A^* \subseteq B^*$ whenever $A \subseteq B$,

(c) $(A^*)^* = A^*$.

For $A, B \in F(D)$ we define the $*$-*multiplication* by $(AB)^* = (A^*B)^* = (A^*B^*)^*$. A fractional ideal $A \in F(D)$ is called a $*$-*ideal* if $A = A^*$ and a $*$-ideal of *finite type* if $A = B^*$ where $B$ is a finitely generated fractional ideal. A star operation $*$ is said to be of *finite character* if $A^* = \bigcup\{B^* \mid 0 \neq B$ is a finitely generated subideal of $A\}$. For $A \in F(D)$ define $A^{-1} = \{x \in K \mid xA \subseteq D\}$ and call $A \in F(D)$ $*$-*invertible* if $(AA^{-1})^* = D$. Clearly every invertible ideal is $*$-invertible for every star operation $*$. If $*$ is of finite character and $A$ is $*$-invertible, then $A^*$ is of finite type. The most well known examples of star operations are: the $v$-operation defined by $A \mapsto A_v = (A^{-1})^{-1}$, the $t$-operation defined by $A \mapsto A_t = \bigcup\{B_v \mid 0 \neq B$ is a finitely generated subideal of $A\}$. By definition $t$ is of finite character. If $*$ is a star operation of finite character then using Zorn's Lemma we can show that a proper integral ideal maximal w.r.t. being a star ideal is a prime ideal and that every proper integral $*$-ideal is contained in a maximal $*$-ideal. Denoting the set of maximal $*$-ideals of $D$ by $* - max(D)$, for a star operation $*$ of finite character, we have $D = \bigcap_{M \in * - \max(D)} D_M$ [19, Proposition 4]. We call $D$ of finite $t$-character if every nonzero nonunit of $D$ belongs to at most a finite number of maximal $t$-ideals of $D$. A $v$-ideal $A$ of finite type is $t$-invertible if and only if $A$ is $t$-locally principal i.e. for every $M \in t - \max(D)$ we have $AD_M$ principal. An integral domain $D$ is called a Prüfer $v$-multiplication domain (PVMD) if every nonzero finitely generated ideal of $D$ is $t$-invertible. According to Griffin [19, Theorem 5] $D$ is a PVMD if and only if $D_M$ is a valuation domain for each $M \in t - \max(D)$. The set $Inv_t(D) = \{A \in F(D) \mid A$ is a $t$-invertible $t$-ideal$\}$ is obviously a group under $t$-multiplication. If we define an order as $A \leq B$ if and only if $A \supseteq B$ then $< Inv_t(D), \leq, \times_t >$ is a directed group [34, Corollar 1.3]. Griffin [19, page 717] with reference to Jaffard [23, page 55] observes that for a PVMD $D < Inv_t(D), \leq, \times_t >$ is a lattice ordered group, see [34, Proposition 2.4] for a direct proof and note that for $A, B \in Inv_t(D)$, $\sup(A, B) = A \cap B$ and $inf(A, B) = (A, B)_t$. Also Griffin proves in [19, Theorem 7] that every nonzero nonunit of a PVMD belongs to at most a finite number of maximal $t$-ideals if and only if $Inv_t(D)$ satisfies Conrad's F-condition: Every positive element is greater than only a finite number of mutually disjoint positive elements. Now as $D$ is the identity of $< Inv_t(D), \leq, \times_t >$, by the definition of order $A \geq D$ implies that $A \subseteq D$. So positive elements of $< Inv_t(D)$,

$\leq, \times_t >$ are precisely the integral $t$-invertible $t$-ideals. Moreover since in a p.o. group $G$ two positive elements are disjoint if $\inf(A,B) =$ identity of $G$, two integral ideals $A, B$ in $Inv_t(D)$ are disjoint if $(A,B)_t = D$ i.e. if $A, B$ are $t$-comaximal. Let $Inv(D)$ denote the group of invertible ideals, let $P(D)$ be the group of nonzero principal fractional ideals of $D$, and let $X \lhd Y$ denote "$X$ is a subgroup of $Y$". Then $P(D) \lhd Inv(D) \lhd Inv_t(D)$. The quotient groups $Inv(D)/P(D)$, $Inv_t(D)/P(D)$ and $Inv_t(D)/Inv(D)$ are respectively the Picard group $Pic(D)$, the $t$-class group $Cl_t(D)$ and the local class group $G(D)$. The Picard group is well known and the other two were introduced in [9].

We say that $D$ is a *$t$-Schreier domain* if $Inv_t(D)$ is a Riesz group. More precisely, $D$ is a $t$-Schreier domain if whenever $A, B_1, B_2$ are $t$-invertible ideals of $D$ and $A \supseteq B_1B_2$, then $A = (A_1A_2)_t$ for some ($t$-invertible) ideals $A_1, A_2$ of $D$ with $(A_i)_t \supseteq B_i$ for $i = 1, 2$. This notion proved useful in the study of factorization of $t$-invertible $t$-ideals with finitely many minimal primes in PVMDs in [15, Lemma 1.8], where it was shown that a PVMD is $t$-Schreier. Since a Prüfer domain is a PVMD, studying $t$-Schreier domains covers a vast area of commutative ring theory of interest.

We show that $D$ is $t$-Schreier if and only if every finite intersection of nonzero principal ideals is a directed union of $t$-invertible $t$-ideals, and consequently every fraction ring of a $t$-Schreier domain is $t$-Schreier. Also, we show that $D$ is a Krull domain if and only if $D$ is a $t$-Schreier Mori domain. (Recall that $D$ is a *Mori domain* if $D$ satisfies ACC on integral divisorial ideals.) We prove that if the polynomial ring $D[X]$ is $t$-Schreier, then $D$ is $t$-Schreier, and the converse is true provided $D$ is integrally closed. For an h-local domain $D$, we prove that $D$ is $t$-Schreier if and only if $D_P$ is $t$-Schreier for every maximal ideal $P$. Next we show that if $D$ is an integrally closed $t$-Schreier domain, $S$ a multiplicative set in $D$, and $X$ an indeterminate over $D_S$, then $D + XD_S[X]$ is $t$-Schreier. We also study $t$-Schreier domains of finite $t$-character and show that a $t$-Schreier domain is of finite $t$-character if and only if every proper integral $t$-invertible $t$-ideal is contained in at most a finite number of mutually $t$-comaximal proper $t$-invertible $t$-ideals if and only if every $t$-locally principal $t$-ideal of $D$ is $t$-invertible. The latter equivalence verifies the now famous Bazzoni Conjecture for $t$-Schreier domains.

Throughout this paper, all rings are commutative and unitary. Our standard reference for any undefined notation or terminology is [18].

## 2. $t$-Schreier domains

We start by linking $t$-Schreier domains with some known concepts.

**Proposition 2.** *Let $D$ be a domain.*

($a$) *$D$ is pre-Schreier if and only if $D$ is $t$-Schreier with zero $t$-class group.*

($b$) *$D$ is quasi-Schreier if and only if $D$ is $t$-Schreier with zero local class group.*

($c$) *The following implications hold*

*$D$ pre-Schreier domain $\Rightarrow$ $D$ quasi-Schreier domain $\Rightarrow$ $D$ $t$-Schreier domain.*

*Proof.* ($a$). By [10, Proposition 1.4], a pre-Schreier domain has zero t-class group, that is, every $t$-invertible $t$-ideal is principal. Clearly, a domain with zero $t$-class group is pre-Schreier if and only if it is t-Schreier.

($b$). Similarly, by [13, Corollary 2.5], a quasi-Schreier domain has zero local class group, that is, every t-invertible t-ideal is invertible. Clearly, a domain with zero local class group is quasi-Schreier if and only if it is t-Schreier.

$(c)$ follows from $(a)$ and $(b)$. •

The following theorem provides the characterizations of $t$-Schreier domains that we consider important for this and future work on this topic.

**Theorem 3.** *Let $D$ be a domain. The following assertions are equivalent:*
$(a)$ *$D$ is a $t$-Schreier domain,*
$(b)$ *every finite intersection of fractional $t$-invertible $t$-ideals is a directed union of fractional $t$-invertible $t$-ideals,*
$(c)$ *every finite intersection of nonzero principal ideals is a directed union of $t$-invertible $t$-ideals,*
$(d)$ *if $a_1,\dots,a_n \in D \setminus \{0\}$ and $I$ is a finitely generated ideal contained in $a_1D \cap \cdots \cap a_nD$, there exists a $t$-invertible $t$-ideal $J$ such that $I \subseteq J \subseteq a_1D \cap \cdots \cap a_nD$.*

*Proof.* Let $K$ be the quotient field of $D$. $(a) \Leftrightarrow (b)$, by Theorem 1. $(b) \Rightarrow (c)$ is obvious. $(c) \Rightarrow (b)$. Every fractional $t$-invertible $t$-ideal $I$ of $D$ is a finite intersection of nonzero fractional principal ideals. Indeed, if $(I(y_1,\dots,y_n))_t = D$ with $y_1,\dots,y_n \in K \setminus \{0\}$, then $I_t = ((y_1,\dots,y_n)_t)^{-1} = (y_1,\dots,y_n)^{-1} = y_1^{-1}D \cap y_2^{-1}D \cap \cdots \cap y_n^{-1}D$. Consequently, every finite intersection of fractional $t$-invertible $t$-ideals can be written as $d^{-1}(a_1D \cap \cdots \cap a_nD)$ with $d, a_1,\dots,a_n \in D \setminus \{0\}$. $(c) \Leftrightarrow (d)$ is clear. •

The following corollary is inspired by [26, Theorem 2].

**Corollary 4.** *Let $D$ be a domain with quotient field $K$. The following assertions are equivalent:*
$(a)$ *$D$ is integrally closed and $t$-Schreier,*
$(b)$ *for every $0 \neq f \in K[X]$, the ideal $fK[X] \cap D[X]$ is the union of ideals $IfD[X]$ where $I$ runs over the set of $t$-invertible fractional subideals of $c(f)^{-1}$, where $c(f)$ denotes the content ideal of $f$.*

*Proof.* $(a) \Rightarrow (b)$. Let $0 \neq f \in K[X]$ and $g \in fK[X] \cap D[X]$. Since $D$ is integrally closed, $g$ belongs to $c(f)^{-1}fD[X]$, cf. [18, Corollary 34.9]. So $g/f \in c(f)^{-1}D[X]$; in particular, $g/f \in K[X]$. Then $J := c(g/f) \subseteq c(f)^{-1}$. $D$ is $t$-Schreier, so part $(b)$ of Theorem 3 implies there exists a $t$-invertible fractional ideal $I$ such that $J \subseteq I \subseteq c(f)^{-1}$. It follows that $g/f \in ID[X]$, hence $g \in IfD[X]$. $(b) \Rightarrow (a)$. We get $fK[X] \cap D[X] = c(f)^{-1}fD[X]$ for each $0 \neq f \in K[X]$, so $D$ is integrally closed, cf. [29, Lemme 1]. Let $0 \neq f \in K[X]$, $J \subseteq c(f)^{-1}$ a finitely generated fractional ideal and let $g \in K[X]$ with $J = c(g)$. Then $gf$ belongs to $fK[X] \cap D[X]$. By hypothesis, there exists a $t$-invertible ideal $I \subseteq c(f)^{-1}$ such that $gf \in IfD[X]$. We get $g \in ID[X]$, so $J \subseteq I \subseteq c(f)^{-1}$. Hence $D$ is $t$-Schreier, by part $(b)$ of Theorem 3. •

**Corollary 5.** *Let $D$ be a $t$-Schreier domain and $a_1,\dots,a_n \in D \setminus \{0\}$. If the ideal $I := a_1D \cap \cdots \cap a_nD$ is $v$-finite, then $I$ is $t$-invertible.*

*Proof.* Assume that $I = (b_1,\dots,b_k)_v$ with $b_1,\dots,b_k \in I \setminus \{0\}$. By part $(d)$ of Theorem 3, there exists a $t$-invertible $t$-ideal $J$ such that $(b_1,\dots,b_k) \subseteq J \subseteq I$. Hence $I = J_t = J$. •

Recall that in [6] and in [2] a domain $D$ was called $v$-coherent if the intersection of every pair of nonzero principal ideals is an ideal of finite type. It so happens that the term *$v$-coherent* has been used by other authors for an apparently more general concept. To be exact a domain $D$ is called $v$-coherent in [17] if the intersection of every pair $A, B$ of $v$-ideals of finite type is of finite type. They also showed [17, Proposition 3.6] that $D$ is $v$-coherent if and only if for each finitely generated ideal $I$ of $D$, $I^{-1}$ is of finite type. In the presence of this usage of "$v$-coherent" we have no choice but to call a domain *$v$-finite conductor* if the intersection of every pair of nonzero principal fractional ideals is a $v$-ideal of finite type, and repair any damage done by the oversight in [6] and in [2]. As we shall see in the sequel, for a $t$-Schreier, quasi Schreier, or for a pre-Schreier domain the property of $D$ being a $v$-finite conductor and that of $D$ being a $v$-coherent domain are equivalent. It is worth a mention also that as far as the authors know there is no example known of a $v$-finite conductor domain that is not $v$-coherent.

For the next corollary let us note that if $I$ is a finitely generated nonzero ideal in a PVMD $D$ then $I^{-1}$ is a $v$-ideal of finite type, because $I$ is $t$-invertible. Also note that $D$ is a Mori domain if and only if for each nonzero ideal $I$ of $D$ there is a finitely generated ideal $F \subseteq I$ such that $F_v = I_v$. So, in particular, $I^{-1}$ is of finite type. In other words, a Mori domain is $v$-coherent.

**Corollary 6.** *Let $D$ be a domain.*

($a$) *The following are equivalent: (i) $D$ is a PVMD, (ii) $D$ is $v$-coherent and $t$-Schreier, (iii) $D$ is $v$-finite conductor and $t$-Schreier.*

($b$) *$D$ is a Krull domain if and only if $D$ is a $t$-Schreier Mori domain.*

*Proof.* ($a$). (i) $\Rightarrow$ (ii) If $D$ is a PVMD, then every finite intersection of nonzero principal ideals is a $t$-invertible ideal, so $D$ is $v$-coherent and $t$-Schreier, by part ($c$) of Theorem 3. (ii) $\Rightarrow$ (iii) is obvious. Finally for (iii) $\Rightarrow$ (i) note that if $D$ is $t$-Schreier, then for every pair $a, b \in D\backslash\{0\}$ the ideal $aD \cap bD$ is $t$-invertible, by Corollary 5. From this it follows that for every pair $a, b \in D\backslash\{0\}$ the ideal $(a, b)$ is $t$-invertible, a necessary and sufficient condition for $D$ to be a PVMD ([24, Corollary 1.8]). ($b$) follows from ($a$), since a Krull domain is precisely a Mori PVMD. •

From part (a) it is evident that for a $t$-Schreier domain (and hence for a quasi Schreier domain or pre-Schreier domain) $v$-finite conductor is equivalent to $v$-coherent. Now since [6] and [2] dealt mainly with quasi-Schreier and $t$-Schreier domains calling a $v$-finite conductor domain $v$-coherent did not have any ill effect.

**Theorem 7.** *Every flat overring (e.g., a fraction ring) of a $t$-Schreier domain is $t$-Schreier.*

*Proof.* Let $D$ be a $t$-Schreier domain and $E$ a flat overring of $D$. Let $I$ be a finite intersection of nonzero principal ideals of $E$. Then $bI = a_1E \cap \cdots \cap a_nE$ for some $a_1,...,a_n, b \in D \setminus \{0\}$. By part ($c$) of Theorem 3, the ideal $J := a_1D \cap \cdots \cap a_nD$ is the union of a directed family $\{Q_j\}_j$ of $t$-invertible $t$-ideals of $D$. Then $I = b^{-1}JE$ is the directed union of the $t$-invertible $t$-ideals $b^{-1}(Q_jE)_t$ of $E$. •

To prepare for the next result we recall from [11] that a prime ideal $P$ of $D$ is called an associated prime of a principal ideal if $P$ is minimal over an ideal of the type $(a) :_D (b)$ for some $a, b \in D$ such that $(0) \neq (a) :_D (b) \neq D$. It was shown in [32, Lemma 6] that if $P$ is an associated prime of a principal ideal of $D$ then $PD_P$ is a prime $t$-ideal of $D_P$.

**Corollary 8.** *Let $D$ be a $t$-Schreier domain. Then for each associated prime $P$ of a principal ideal of $D$ the localization $D_P$ is a pre-Schreier domain.*

*Proof.* $D_P$ is $t$-Schreier by Theorem 7. But as $P$ is an associated prime of a principal ideal $PD_P$ is a $t$-ideal. So every $t$-invertible $t$-ideal of $D_P$ is principal. So any statement involving $t$-invertible $t$-ideals is about principal ideals and combining this fact with $D_P$ being $t$-Schreier we have the result. □

**Theorem 9.** *Let $D$ be an integrally closed integral domain.*
($a$) *If the polynomial ring $D[X]$ is $t$-Schreier, then $D$ is $t$-Schreier.*
($b$) *If $D$ is $t$-Schreier, then $D[X]$ is $t$-Schreier.*

*Proof.* ($a$) Let $a_1,...,a_n \in D \setminus \{0\}$ and $J$ a nonzero finitely generated ideal of $D$ contained in $a_1D \cap \cdots \cap a_nD$. As $D[X]$ is $t$-Schreier, there exists a $t$-invertible $t$-ideal $I$ of $D[X]$ such that $J \subseteq I \subseteq a_1D[X] \cap \cdots \cap a_nD[X]$. Because $D$ is integrally closed $I = QD[X]$ for some $t$-invertible $t$-ideal $Q$ of $D$ [3, Theorem 3.1]. Contracting to $D$ we get $J \subseteq Q \subseteq a_1D \cap \cdots \cap a_nD$.

($b$) Assume that $D$ is an integrally closed $t$-Schreier domain. Let $I$ be a finite intersection of nonzero principal ideals of $D[X]$. As remarked at the end of the proof of [2, Theorem 9] there exist $a_1,...,a_n \in D$ and $f, g \in D[X] \setminus \{0\}$ such that $fI = g(a_1D[X] \cap \cdots \cap a_nD[X])$. By part ($c$) of Theorem 3, the ideal $J := a_1D \cap \cdots \cap a_nD$ is the union of a directed family $\{Q_j\}_j$ of $t$-invertible $t$-ideals of $D$. Then $I = (g/f)JD[X]$ is the directed union of the $t$-invertible $t$-ideals $(g/f)Q_jD[X]$ of $D[X]$. •

**Corollary 10.** *Let $D$ be an integrally closed $t$-Schreier domain and let $S \subseteq D$ a multiplicative set. Then $D + XD_S[X]$ is a $t$-Schreier domain.*

*Proof.* Set $E = D + XD_S[X]$ and $K =$ the quotient field of $D$. Let $f_1, ...f_n \in E - \{0\}$ and $g_1, ..., g_m \in \cap_i f_iE$. Since $E = \cup_{s\in S} D[X/s]$ we can assume that $f_i$,$g_j \in D[X]$ and $g_j \in \cap_i f_iD[X]$. $D$ is integrally closed, so as argued in the proof of part ($b$) of Theorem 9, there exist $0 \neq w \in K(X)$ and $a_1, ..., a_p \in D$ such that $w(\cap_i f_iD[X]) = \cap_k a_kD[X]$. So $w(g_1, ..., g_m)D[X] \subseteq \cap_k a_kD[X]$. As $D$ is $t$-Schreier, $c(wg_1) + \cdots + c(wg_m) \subseteq I \subseteq \cap_k a_kD$ for some $t$-invertible ideal $I$ of $D$, where $c(wg_j)$ is the content ideal of $wg_j$. We get $(g_1, ..., g_m)D[X] \subseteq w^{-1}ID[X] \subseteq w^{-1}(\cap_k a_kD[X]) = \cap_i f_iD[X]$, hence $(g_1, ..., g_m)E \subseteq w^{-1}IE \subseteq \cap_i f_iE$. Finally, since $E$ is a flat $D$-module, $w^{-1}IE$ is a $t$-invertible ideal of $E$. •

Let $D$ be a domain and $\mathcal{F}$ a family of nonzero prime ideals of $D$. According to [5] $\mathcal{F}$ is called an *IFC family* if $D = \cap_{P\in\mathcal{F}} D_P$, every nonzero $x \in D$ belongs to only finitely many members of $\mathcal{F}$, and if no two members of $\mathcal{F}$ contain a nonzero prime ideal of $D$. We say that $D$ is an *$\mathcal{F}$-IFC domain* if $D$ has an IFC family $\mathcal{F}$. The class of $\mathcal{F}$-IFC domains includes the $h$-local domains of Matlis [25], Noetherian domains whose grade-one primes are of height one, Krull domains, the generalized Krull domains of Ribemboim [30] and independent rings of Krull type of Griffin [20]. Let $\mathcal{F}$ be an IFC family. The mapping $I \mapsto I^* = \cap_{P\in\mathcal{F}}(ID_P)_t : F(D) \to F(D)$ is a finite character star operation on $D$.

**Theorem 11.** *An $\mathcal{F}$-IFC domain $D$ is $t$-Schreier if and only if $D_P$ is $t$-Schreier for every $P \in F$.*

*Proof.* The "only if" follows from Theorem 7. For the "if" part, let $I \supseteq AB$ with $I, A, B$ $t$-invertible $t$-ideals of $D$ and let $M_1, M_2, ..., M_n$ be the members of $\mathcal{F}$ containing $I$. For each $i$, set $D_i = D_{M_i}$, $I_i = ID_i$, $A_i = AD_i$, $B_i = BD_i$. Clearly, $I_i$, $A_i$, $B_i$ are $t$-invertible $t$-ideals of $D_i$ and $I_i \supseteq A_iB_i$. Since $D_i$ is $t$-Schreier, $I_i = (A_i''B_i'')_t$ for some t-invertible t-ideals $A_i'', B_i''$ of $D_i$ with $A_i'' \supseteq A_i$ and $B_i'' \supseteq B_i$. Since $D$ is an $\mathcal{F}$-IFC-domain, the only member of $\mathcal{F}$ which can contain $A_i' = A_i'' \cap D$ (resp. $B_i' = B_i'' \cap D$) is $M_i$, cf. [5, Lemma 2.3]. Let $*$ be the star operation on $D$ given by $I \mapsto I^* = \cap_{P \in \mathcal{F}} (ID_P)_t$ (see the paragraph preceding Theorem 11). Set $A' = A_1'A_2' \cdots A_n'$ and $B' = B_1'B_2' \cdots B_n'$. For $i = 1, ..., n$, we get $((AB)_{M_i})_t = (A_i'B_i'D_{M_i})_t = (I_i)_t$, so $I^* = (A'B')^*$. Hence $I = I_t = (A'B')_t$, because $*$ is a star operation of finite type cf. [1, Theorem 2]. Similarly, we get that ${A'}_t \supseteq A_t$ and ${B'}_t \supseteq B_t$. Thus $D$ is $t$-Schreier. •

## 3. $t$-Schreier domains of finite $t$-character

In this section we use a modification of the approach of [28] in order to characterize the $t$-Schreier domains of finite $t$-character. As a consequence of this work we can conclude that every nonzero element of a Prüfer domain $D$ belongs to a finite number of maximal ideals of $D$ if and only if every nonzero finitely generated ideal of $D$ is contained in at most a finite number of mutually comaximal proper invertible ideals. The following result extends [28, Proposition 2.1].

**Proposition 12.** *Let $D$ be a $t$-Schreier domain and $x_1, ..., x_n \in D \setminus \{0\}$ such that $(x_1, ..., x_n)_v \neq D$. Then there exists a $t$-invertible $t$-ideal $H$ such that $(x_1, ..., x_n) \subseteq H \neq D$.*

*Proof.* Since $(x_1, ..., x_n)_v \neq D$, we have $(x_1, ..., x_n) \subseteq \frac{a}{b}D \cap D$, for some $a, b \in D \setminus \{0\}$ with $a$ not dividing $b$, cf. [34, Observation A, page 432]. By Theorem 3, there exists a $t$-invertible $t$-ideal $H$ such that $(x_1, ..., x_n) \subseteq H \subseteq \frac{a}{b}D \cap D \subset D$. •

Following [28] we call a $t$-invertible $t$-ideal $A \neq D$ *homogeneous* if $A \neq D$ and for all $t$-invertible $t$-ideals $H, K \neq D$ with $H, K \supseteq A$ we have $(H, K)_t \neq D$. The following result is an extension of [28, Proposition 2.1].

**Proposition 13.** *Let $D$ $t$-Schreier domain and $A$ a proper $t$-invertible ideal of $D$. Then $A$ is homogeneous if and only if $A$ is contained in a unique maximal $t$-ideal.*

*Proof.* ($\Leftarrow$). Suppose that $A$ is contained in a unique maximal $t$-ideal $M$ and let $H, K$ be proper $t$-invertible $t$-ideals containing $A$. Then $H, K \subseteq M$, so $(H, K)_t \subseteq M$. ($\Rightarrow$). Suppose that $A$ is contained in two distinct maximal $t$-ideals $M_1, M_2$. Hence $(M_1, M_2)_t = D$, so we can choose finitely generated ideals $F_i \subseteq M_i$, $i = 1, 2$, such that $A \subseteq (F_i)_t$ and $(F_1, F_2)_t = D$. By Proposition 12, there exist proper $t$-invertible $t$-ideals $G_i$ such that $F_i \subseteq G_i$, $i = 1, 2$. Hence $A \subseteq G_1, G_2$ and $(G_1, G_2)_t = D$, so $A$ is not homogeneous. •

Recall that a domain $D$ is said to be of *finite $t$-character* if every nonzero element of $D$ belongs to finitely many maximal $t$-ideals. Recently the authors [14] have used the property of $t$-Schreier domains given in Proposition 12 to call an integral domain $D$ $*$-sub-Prüfer, for a finite character star operation $*$, if every nonzero finitely generated ideal $I$ of $D$ with $I^* \neq D$ is contained in a $*$-invertible $*$-ideal of $D$. The results that follow are consequences of considerations in [14]. But since [14] is very compact and essentially proves only one very general theorem it would be useful to go through the proofs of the following results, step by step.

**Theorem 14.** *For a $t$-Schreier domain $D$, the following assertions are equivalent:*
($a$) *$D$ is of finite $t$-character.*
($b$) *Every infinite family of proper mutually $t$-comaximal $t$-invertible $t$-ideals has zero intersection.*

*Proof.* The implication $(a) \Rightarrow (b)$ is clear since a maximal $t$-ideal cannot contain two $t$-comaximal $t$-ideals. Conversely, assume that $(b)$ holds. Denote by $\Gamma$ the set of proper $t$-invertible $t$-ideals of $D$. First we show the following property: ($\sharp$) every ideal in $W \in \Gamma$ is contained in some homogeneous ideal. Deny. As $W$ is not homogeneous, there exist $P_1, N_1 \in \Gamma$ containing $W$ such that $(P_1, N_1)_t = D$. Since $N_1$ is not homogeneous, there exist $P_2, N_2 \in \Gamma$ containing $N_1$ such that $(P_2, N_2)_t = D$. Note that $(P_1, P_2)_t = (P_1, N_2)_t = D$. By induction, we can construct an infinite sequence $(P_k)_{k \geq 1}$ of mutually $t$-comaximal ideals in $\Gamma$ with $W \subseteq P_k$, $k \geq 1$. This fact contradicts condition $(b)$. So ($\sharp$) holds.

Now, let $I$ be a proper $t$-invertible $t$-ideal of $D$. By ($\sharp$), $I$ is contained in some homogeneous ideal $H_1$. Assume there is some $J_1 \in \Gamma$ containing $I$ with $(J_1, H_1)_t = D$. By ($\sharp$), $J_1$ is contained in some homogeneous ideal $H_2$. So $H_1, H_2$ are $t$-comaximal. Assume there is some $J_2 \in \Gamma$ containing $I$ with $(J_2, H_i)_t = D$, $i = 1, 2$. By ($\sharp$), $J_2$ is contained in some homogeneous ideal $H_3$. So $H_1, H_2, H_3$ are mutually $t$-comaximal. We can continue in this way. By $(b)$, this process has to stop after a finite number of steps. At that moment we find a finite set $H_1, ..., H_n \in \Gamma$ of mutually $t$-comaximal homogeneous ideals containing $I$ such that there is no $J \in \Gamma$ containing $I$ with $(J, H_i)_t = D$, $1 \leq i \leq n$. Let $M$ be a maximal $t$-ideal containing $I$ and $0 \neq x \in M$. Then $(I, x)$ is not $t$-comaximal with some ideal $H_j$. Since $H_j$ is homogeneous, we get $x \in (I, x) \subseteq M_j$, where $M_j$ the unique maximal t-ideal containing $H_j$ (cf. Lemma 13). So $M \subseteq M_1 \cup \cdots \cup M_n$. By the Prime Avoidance Lemma, $M$ is contained in some $M_k$, hence $M = M_k$. Thus $M_1, ..., M_n$ are the maximal $t$-ideals containing $I$. Hence $(a)$ holds. •

We recall that Griffin [20] called a PVMD of finite $t$-character a *ring of Krull type*. Since a PVMD is a $t$-Schreier domain we have the following corollary.

**Corollary 15.** *A PVMD is a ring of Krull type if and only if every infinite family of proper mutually $t$-comaximal $t$-invertible $t$-ideals has zero intersection.*

Next, we adapt a recent result proved by the second author [35] for PVMDs to show that for a $t$-Schreier domain $D$ to be of finite $t$-charater it is enough to show that every $t$-locally principal ideal is $t$-invertible. We need the following lemma. Let us first recall that if $\{M_\alpha\}$ is the set of all maximal $t$-ideals of $D$, then the operation $A \mapsto A_w = \cap AD_{M_\alpha}$ is a star operation of finite character and so $w \leq t$, i.e. $(A_w)_t = A_t$.

**Lemma 16.** *If $A_1, A_2, ..., A_n$ are mutually $t$-comaximal ideals then $(A_1 \cap A_2 \cap ... \cap A_n)_t = (A_1 A_2 ... A_n)_t$.*

Proof. Since $A_i$ are mutually $t$-comaximal, they do not share a maximal $t$-ideal. Now if $M$ is a maximal $t$-ideal that does not contain any of $A_i$ then $D_M =$

$(A_1 \cap A_2 \cap ... \cap A_n)D_M = (A_1A_2...A_n)D_M$. If on the other hand $M$ is a maximal $t$-ideal that contains at least one and hence exactly one of them, say $A_i$ then $A_iD_M = (A_1 \cap A_2 \cap ... \cap A_n)D_M = (A_1A_2...A_n)D_M$. Thus $(A_1 \cap A_2 \cap ... \cap A_n)_w = (A_1A_2...A_n)_w$, but this means $(A_1 \cap A_2 \cap ... \cap A_n)_t = (A_1A_2...A_n)_t$. •

**Proposition 17.** *A $t$-Schreier domain $D$ is of finite $t$-character if and only if every $t$-locally principal $t$-ideal of $D$ is $t$-invertible.*

*Proof.* If $D$ is of finite $t$-character, then every $t$-locally principal ideal is $t$-invertible, cf. [4, Lemma 2.2]. Conversely, assume that $D$ is not of finite $t$-character. We shall construct a $t$-locally principal ideal which is not t-invertible. By Theorem 14, there is an integral $t$-invertible $t$-ideal $A$ and an infinite sequence $(H_n)_{n\geq 1}$ of proper mutually $t$-comaximal $t$-invertible $t$-ideals such that $A$ is contained in every $H_n$. Let $x \in A\backslash\{0\}$. By Lemma 16, $x \in (H_1 \cdots H_n)_t$ for each $n \geq 1$, so $(H_1^{-1} \cdots H_n^{-1})_t x \subseteq D$. We get the ascending sequence of $t$-ideals

$$(H_1^{-1})_t x \subset (H_1^{-1}H_2^{-1})_t x \subset ... \subset (H_1^{-1} \cdots H_n^{-1})_t x \subset ... \ .$$

Each inclusion is strict since $(H_1^{-1} \cdots H_n^{-1})_t x = (H_1^{-1} \cdots H_n^{-1}H_{n+1}^{-1})_t x$ implies $H_{n+1} = D$, because every $H_i$ is $t$-invertible. So $F = \cup_{n\geq 1}(H_1^{-1} \cdots H_n^{-1})_t x$ is a $t$-ideal but is not of finite type, hence is not $t$-invertible. We show that $F$ is $t$-locally principal. Let $M$ be a maximal $t$-ideal of $D$. If $M$ contains no $H_n$, then $FD_M = xD_M$. Otherwise, $M$ contains exactly one $H_n$, say $H_{n_0}$, because the ideals $H_n$ are mutually $t$-comaximal. In this latter case, as $H_{n_0}$ is $t$-invertible, we get that $FD_M = xH_{n_0}^{-1}D_M = x(H_{n_0}D_M)^{-1}$ is principal. So $F$ is a $t$-locally principal ideal which is not $t$-invertible. •

Note that since a PVMD is $t$-Schreier and since a Prüfer domain is a PVMD in which every nonzero ideal is a $t$-ideal, every result proved for $t$-Schreier domains involving the $t$-operation is a result proved for PVMDs involving the $t$-ideals and is a result for Prüfer domains involving nonzero ideals. So we have the following corollary.

**Corollary 18.** *A PVMD (resp., a Prüfer domain) $D$ is of finite $t$-character (respectively, of finite character) if and only if every $t$-locally principal (respectively, locally principal) nonzero ideal of $D$ is $t$-invertible (resp., invertible).*

Bazzoni conjectured in [7] and [8] that if $D$ is a Prüfer domain such that every locally principal ideal is finitely generated then $D$ is of finite character. Part of Corollary 18 resolves Bazzoni's conjecture, hopefully, in a manner simpler than earlier verifications of her conjecture by Holland et al. [22], McGovern [27] and Franz Halter-Koch [21]. Franz Halter-Koch worked with specializations of PVMDs called $*$-Prüfer domains for $*$ of finite character, using the language of monoids and of ideal systems. As it was indicated in [35] it is sufficient, in this case, to deal with PVMDs. Our work extends the work done for PVMDs, in [35], to $t$-Schreier domains. However remaining within $t$-Schreier domains we cannot cover the more general approach of [35].

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