T-SPLITTING MULTIPLICATIVE SETS OF IDEALS IN INTEGRAL DOMAINS

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Abstract. Let D be an integral domain. We study those multiplicative sets of ideals $\mathcal{S}$ of D with the property that every nonzero principal ideal dD of D can be written as $dD = (AB)_t$ with A, B ideals of D such that A contains some ideal in $\mathcal{S}$ and $(C + B)_t = D$ for each $C \in \mathcal{S}$.

Let D be an integral domain with quotient field K and let $F(D)$ be the set of nonzero fractional ideals of D. Clearly, for $A \in F(D)$, $A^{-1} = D :_K A$ is again in $F(D)$. Recall that a mapping $A \mapsto A^*$ of $F(D)$ into itself is called a **star operation** on D if the following conditions hold for all $a \in K \setminus \{0\}$ and $A, B \in F(D)$: (1) $(aD)^* = aD$, $(aA)^* = aA^*$, (2) $A \subseteq A^*$, if $A \subseteq B$, then $A^* \subseteq B^*$, and (3) $(A^*)^* = A^*$. A is a $*$-ideal if $A = A^*$. For standard material about star operations, see sections 32 and 34 of [9]. Three well-known examples of star operations are the maps $A \mapsto A$ (the **d-operation**), $A \mapsto A_v$ (the **v-operation**) and $A \mapsto A_t$ (the **t-operation**), where $A_v = (A^{-1})^{-1}$ and $A_t = \cup\{B_v |\, 0 \neq B \subseteq A \text{ is finitely generated}\}$. Clearly, $A_v = A_t$ if A is finitely generated. An ideal $A \in F(D)$ is **t-invertible** if $(AA^{-1})_t = D$. In this case A has finite type, that is, $A_t = (x_1, ..., x_n)_t$ for some $x_1, ..., x_n \in A$. D is called a **Prüfer v-multiplication domain** (PVMD), if every finitely generated ideal $A \in F(D)$ is t-invertible. The t-class group $Cl_t(D)$ of D is the group of t-invertible fractional t-ideals, under the product $A * B = (AB)_t$, modulo its subgroup of principal fractional ideals.

The following concept was introduced and studied in [3]. A multiplicative subset S of D is said to be **t-splitting**, if for each $d \in D \setminus \{0\}$, $dD = (AB)_t$ for some ideals A, B of D with $A_t \cap S \neq \emptyset$ and $(B, s)_t = D$ for each $s \in S$. The main result of [3] asserts that $D + XD_S[X]$ is a PVMD if and only if D is a PVMD and S is a t-splitting set of D, where $D + XD_S[X]$ is the subring of $D_S[X]$ consisting of those $f \in D_S[X]$ with constant term in D. The t-splitting sets are investigated further in [6].

The main purpose of this note is to extend certain results from [3] and [6] to the case of multiplicative sets of ideals. We aim to show that by using the notion of t-splitting sets of ideals, we can explain a number of multiplicative phenomena that cannot be explained otherwise or are hard to explain. The main concept we use is that of a t-splitting set of ideals $\mathcal{S}$ of a domain D (see Definition 1). We show that many results from [3] and [6] can be stated for t-splitting sets of ideals. A characterization of $\mathcal{S}$ being t-splitting using the $\mathcal{S}$-transform of D (see definition below) is given in Proposition 5. In Theorem 12, we show that the presence of a t-splitting set of ideals induces a natural cardinal product decomposition of the ordered monoid of fractional t-ideals of D (with the t-product and ordered by reverse

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inclusion). Restricting to t-prime ideals, this decomposition gives a well-behaved partition of the set of t-prime (resp. t-maximal) ideals of D (see Remark 14 and Corollary 15). Some applications for PVMDs and Krull domains are given in Propositions 16 and 17. The final part of this note contains several Nagata-type theorems.

Throughout this note, all rings are integral domains. All undefined terminology is standard as in [9]. Let D be an integral domain with quotient field K, $\mathcal{S}$ a multiplicative set of ideals of D and $D_{\mathcal{S}} = \{x \in K |\ xA \subseteq D \text{ for some } A \in \mathcal{S}\}$ the $\mathcal{S}$-transform of D (see [4] for basic properties of this construction). If I is an ideal of D, then $I_{\mathcal{S}} = \{x \in K |\ xA \subseteq I \text{ for some } A \in \mathcal{S}\}$ is an ideal of $D_{\mathcal{S}}$ containing I. Denote by $\mathcal{S}^{\perp}$ the set of all ideals B of D with $(A+B)_t = D$ for all $A \in \mathcal{S}$. Note that $\mathcal{S}^{\perp}$ is also a multiplicative set of ideals. Call it the **t-complement** of $\mathcal{S}$. Consider also, the multiplicative set of ideals $sp(\mathcal{S}) \supseteq \mathcal{S}$ consisting of all ideals C of D with $C_t \supseteq A$ for some $A \in \mathcal{S}$. It is easy to see that $sp(sp(\mathcal{S})) = sp(\mathcal{S})$, $sp(\mathcal{S})^{\perp} = \mathcal{S}^{\perp}$ and $D_{\mathcal{S}} = D_{sp(\mathcal{S})}$.

We begin by providing a formal definition of the notion of t-splitting sets of ideals.

Definition 1. Let $\mathcal{S}$ be a multiplicative set of ideals of D and $\mathcal{S}^{\perp}$ its t-complement. We call $\mathcal{S}$ a **t-splitting set of ideals** if every nonzero principal ideal dD of D can be written as $dD = (AB)_t$ with $A \in sp(\mathcal{S})$ and $B \in \mathcal{S}^{\perp}$.

Clearly, $\mathcal{S}$ is t-splitting if and only if $sp(\mathcal{S})$ is t-splitting. If $S \subseteq D$ is a saturated multiplicative set of D and $\mathcal{S} = \{sD |\ s \in S\}$, then S is t-splitting in the sense of [3] if and only if $\mathcal{S}$ is t-splitting in our sense.

In a Krull domain E, every nonzero proper principal ideal can be (uniquely) written as a t-product of height-one primes [7, Theorem 3.12], so every set of height-one prime ideals of E generates a t-splitting set (see also Proposition 17). Some easy consequences of Definition 1 are given below.

Proposition 2. If $\mathcal{S}$ is a t-splitting set of ideals of D, then the following assertions hold.

(a) $\mathcal{S}^{\perp}$ is t-splitting.

(b) For every $C \in \mathcal{S}$, C_t contains some t-invertible ideal of $sp(\mathcal{S})$.

(c) The set $\mathcal{S}_i$ of all t-invertible ideals in $sp(\mathcal{S})$ is a t-splitting set with t-complement $\mathcal{S}^{\perp}$ and $sp(\mathcal{S}_i) = sp(\mathcal{S})$.

Proof. (a) is clear from Definition 1. For (b) and (c), note that when $0 \neq d \in C \in \mathcal{S}$ and $dD = (AB)_t$ with $A \in sp(\mathcal{S})$ and $B \in \mathcal{S}^{\perp}$, it follows that A is t-invertible and $C_t \supseteq A$. Indeed, as $C \in \mathcal{S}$ and $B \in \mathcal{S}^{\perp}$, we get $(C+B)_t = D$, so $A \subseteq A_t = (A(C+B))_t \subseteq C_t$. So, (b) follows, and, consequently, $sp(\mathcal{S}_i) \supseteq sp(\mathcal{S})$. Thus (c) follows from the remarks accompanying Definition 1. ¤

In [8], a multiplicative set of ideals $\mathcal{S}$ of D is said to be **v-finite** if for each $A \in \mathcal{S}$, A_t contains some v-finite ideal $J \in sp(\mathcal{S})$. Since an invertible t-ideal is v-finite, part (b) of the preceding result shows that a t-splitting set is v-finite. Our next result shows that, when $\mathcal{S}$ is t-splitting, the t-product decomposition imposed in Definition 1 for the principal ideals extends to all t-ideals (thus extending [3, Lemma 4.6]).

Proposition 3. Let $\mathcal{S}$ be a t-splitting set of ideals of D. Then for every nonzero ideal I of D, I_t can be written as $I_t = (AB)_t$ with $A \in sp(\mathcal{S})$ and $B \in \mathcal{S}^{\perp}$.

This decomposition is unique in the following sense. If $(AB)_t = (A'B')_t$ with $A, A' \in sp(\mathcal{S})$ and $B, B' \in \mathcal{S}^{\perp}$, then $A_t = A'_t$ and $B_t = B'_t$. In particular, if I_t is of finite type, then we can choose A and B to be finite type t-ideals.

Proof. Let I be a nonzero ideal of D and set $J = I\setminus\{0\}$. As $\mathcal{S}$ is a t-splitting set, for each $j \in J$, we can write $jD = (A_jB_j)_t$ with $A_j \in sp(\mathcal{S})$ and $B_j \in \mathcal{S}^{\perp}$. Then $I_t = (\sum_j jD)_t = (\sum_j (A_jB_j)_t)_t = (\sum_j A_jB_j)_t$. But $(\sum_j A_jB_j)_t = ((\sum_h A_h)(\sum_i B_i))_t$. Indeed, the inclusion $\subseteq$ is clear. For $\supseteq$, let $h, i \in J$, $h \neq i$. Then $(A_i + B_h)_t = D$, so $A_hB_i \subseteq (A_hB_i(A_i + B_h))_t \subseteq (\sum_j A_jB_j)_t$. Finally, note that $\sum_j A_j \in sp(\mathcal{S})$ and $\sum_j B_j \in \mathcal{S}^{\perp}$.

For the uniqueness part, assume that $(AB)_t = (A'B')_t$ with $A, A' \in sp(\mathcal{S})$ and $B, B' \in \mathcal{S}^{\perp}$. Since $(A + B')_t = (A' + B)_t = D$, we get $A_t = (A(A' + B))_t = (AA' + (AB)_t)_t = (AA' + (A'B')_t)_t = ((A + B')A')_t = A'_t$. Similarly, $B_t = B'_t$.

The "in particular" part was proved on the way. ¤

As a consequence, $\mathcal{S}^{\perp\perp} = sp(\mathcal{S})$. Indeed, let C be in the t-complement of $\mathcal{S}^{\perp}$. As shown above, $C_t = (AB)_t$ for some $A \in sp(\mathcal{S})$ and $B \in \mathcal{S}^{\perp}$. Since $(C+B)_t = D$ and $C \subseteq B_t$, we get $B_t = D$. So $C_t = A_t \in sp(\mathcal{S})$, hence $C \in sp(\mathcal{S})$.

In Proposition 5, we generalize [3, Lemma 4.2]. We need the next lemma which relies on [14, Lemma 3.4] and [8, Proposition 1.2].

Lemma 4. Let $\mathcal{S}$ be a multiplicative set of ideals of D and I a nonzero ideal of D. Then

(a) $(ID_{\mathcal{S}})_t = (I_tD_{\mathcal{S}})_t$.

(b) If I is a t-invertible ideal of D and $(ID_{\mathcal{S}})_t = D_{\mathcal{S}}$, then $I \in sp(\mathcal{S})$.

Proof. (a) is a part of [14, Lemma 3.4]. For (b), assume that I is t-invertible. By [8, Proposition 1.2], $(JD_{\mathcal{S}})_t = (J_t)_{\mathcal{S}}$ for each finitely generated nonzero ideal J of D with $D : J$ v-finite. As I is t-invertible, $I_t = J_t$ for some finitely generated ideal $J \subseteq I$. Moreover, $D : I = D : J$ is v-finite and, by (a), $(ID_{\mathcal{S}})_t = (JD_{\mathcal{S}})_t$. So, $D_{\mathcal{S}} = (ID_{\mathcal{S}})_t = (JD_{\mathcal{S}})_t = (J_t)_{\mathcal{S}} = (I_t)_{\mathcal{S}}$. Hence $1 \in (I_t)_{\mathcal{S}}$, that is, $H \subseteq I_t$ for some $H \in \mathcal{S}$. Consequently, $I \in sp(\mathcal{S})$. ¤

Proposition 5. Let $\mathcal{S}$ be a multiplicative set of ideals of D. Then $\mathcal{S}$ is t-splitting if and only if $\mathcal{S}$ is v-finite and $dD_{\mathcal{S}} \cap D$ is a t-invertible ideal for each $0 \neq d \in D$.

Proof. Assume that $\mathcal{S}$ is t-splitting. Then $\mathcal{S}$ is v-finite, as shown in the paragraph after Proposition 2. Let $0 \neq d \in D$. Then $dD = (AB)_t$ for some $A \in \mathcal{S}$ and $B \in \mathcal{S}^{\perp}$. As B is t-invertible, it suffices to show that $dD_{\mathcal{S}} \cap D = B_t$. In particular, it will follow that $dD_{\mathcal{S}} \cap D \in \mathcal{S}^{\perp}$. As $A(d^{-1}B_t) \subseteq d^{-1}(AB)_t = D$, we get $d^{-1}B_t \subseteq D_{\mathcal{S}}$, hence $B_t \subseteq dD_{\mathcal{S}} \cap D$. On the other hand, let $x \in dD_{\mathcal{S}} \cap D$. Then $C(d^{-1}x) \subseteq D$ for some $C \in \mathcal{S}$. So $Cx \subseteq dD \subseteq B_t$, hence $x \in B_t$, because $(C + B)_t = D$.

Conversely, assume that $\mathcal{S}$ is v-finite and $dD_{\mathcal{S}} \cap D$ is a t-invertible ideal for each $0 \neq d \in D$. Let $0 \neq d \in D$. As $B = dD_{\mathcal{S}} \cap D$ is a t-invertible ideal containing dD, $dD = (AB)_t$ for some (t-invertible) ideal A of D. Note that $BD_{\mathcal{S}} \subseteq dD_{\mathcal{S}}$. By part (a) of Lemma 4, we get $dD_{\mathcal{S}} = ((AB)_tD_{\mathcal{S}})_t = (ABD_{\mathcal{S}})_t \subseteq (dAD_{\mathcal{S}})_t$, hence $(AD_{\mathcal{S}})_t = D_{\mathcal{S}}$. By part (b) of Lemma 4, $A \in sp(\mathcal{S})$. To verify that $B \in \mathcal{S}^{\perp} = sp(\mathcal{S})^{\perp}$, it suffices to show that $(B + H)_t = D$ for each t-ideal $H \in sp(\mathcal{S})$. By the second part of our assumption, we may assume that H is v-finite. If $x \in H^{-1} \cap B^{-1}$, then $x \in D_{\mathcal{S}}$, so $Bx \subseteq BD_{\mathcal{S}} \cap D = dD_{\mathcal{S}} \cap D = B$. As B is t-invertible, $x \in D$. Thus

$(H+B)^{-1} = H^{-1} \cap B^{-1} = D$, that is, $(H+B)_v = D$. So $(H+B)_t = (H+B)_v = D$, because H and B are v-finite ideals. Thus $B \in \mathcal{S}^{\perp}$. ¤

To see that in the 'if' part of the preceding proposition, the assumption that $\mathcal{S}$ is v-finite is essential, we may use the following example from [8]. Let V be a nontrivial valuation domain whose maximal ideal M is idempotent and $\mathcal{S} = \{D, M\}$. Then $V_{\mathcal{S}} = V$, because $V : M = V$. So $dV_{\mathcal{S}} \cap V$ is t-invertible for each $0 \neq d \in V$. However, $\mathcal{S}$ is not v-finite.

Remark 6. Let $\mathcal{S}$ be a t-splitting set of ideals of D, I a nonzero ideal of $D_{\mathcal{S}}$ and $0 \neq d \in I \cap D$. As shown in the proof of Proposition 5, $dD_{\mathcal{S}} \cap D \in \mathcal{S}^{\perp}$. Hence $I \cap D \in \mathcal{S}^{\perp}$, because $I \cap D \supseteq dD_{\mathcal{S}} \cap D$. Similarly, $I \cap D \in sp(\mathcal{S})$ whenever I is a nonzero ideal of $D_{\mathcal{S}^{\perp}}$.

The next proposition is only a restatement, in our setup, of [3, Theorem 4.10]. The proof is virtually the same. We begin with a simple lemma.

Lemma 7. If $\mathcal{S}$ is a multiplicative set of ideals of D, then $D = D_{\mathcal{S}} \cap D_{\mathcal{S}^{\perp}}$.

Proof. Let $x \in D_{\mathcal{S}} \cap D_{\mathcal{S}^{\perp}}$. Then $xA \subseteq D$ and $xB \subseteq D$ for some $A \in \mathcal{S}$ and $B \in \mathcal{S}^{\perp}$. So $xD = x(A+B)_t = (xA+xB)_t \subseteq D$, hence $x \in D$. ¤

Proposition 8. Let $\mathcal{S}$ be a t-splitting set of ideals of D and I a nonzero ideal of D. Then

$$I_t = (ID_{\mathcal{S}})_t \cap (ID_{\mathcal{S}^{\perp}})_t = (((ID_{\mathcal{S}})_t \cap D)((ID_{\mathcal{S}^{\perp}})_t \cap D))_t.$$

Proof. By Lemma 7, $D = D_{\mathcal{S}} \cap D_{\mathcal{S}^{\perp}}$. Hence by [1, Theorem 2], the map sending a nozero fractional ideal A of D into $A^* = (AD_{\mathcal{S}})_t \cap (AD_{\mathcal{S}^{\perp}})_t$ is a finite character star-operation on D. Consequently, $I_t \supseteq I^*$. Part (a) of Lemma 4 supplies the opposite inclusion. For the second equality, set $U = (ID_{\mathcal{S}})_t \cap D$ and $V = (ID_{\mathcal{S}^{\perp}})_t \cap D$. By Remark 6, $U \in \mathcal{S}^{\perp}$ and $V \in sp(\mathcal{S})$, so $(U+V)_t = D$. Consequently, $I_t = U \cap V = (U \cap V)_t = (UV)_t$. ¤

Remark 9. Let $\mathcal{S}$ be a t-splitting set of ideals of D and I a nonzero ideal of D. By Proposition 3, $I_t = (AB)_t$ with $A \in sp(\mathcal{S})$ and $B \in \mathcal{S}^{\perp}$. Combining the previous result, Remark 6 and Proposition 3, we get $A_t = (ID_{\mathcal{S}^{\perp}})_t \cap D$ and $B_t = (ID_{\mathcal{S}})_t \cap D$. Note that $(ID_{\mathcal{S}})_t \cap D$ and $(ID_{\mathcal{S}^{\perp}})_t \cap D$ are t-ideals of D, cf. Lemma 4 and [5, Proposition 1.1].

Let D be a domain. By definition, a t-prime ideal of D is a nonzero prime ideal of D which is also a t-ideal. It is well-known that a prime ideal which is minimal over a nonzero principal ideal is t-prime. Also, a maximal t-ideal, that is, a maximal element of the set of all proper t-ideals, is a t-prime ideal (see e.g. [12]).

Proposition 10. Let $\mathcal{S}$ be a t-splitting set of ideals of D with t-complement $\mathcal{S}^{\perp}$ and let P be a prime t-ideal of D. Then P is either in $sp(\mathcal{S})$ or in $\mathcal{S}^{\perp}$. Moreover, if $P \in \mathcal{S}^{\perp}$ and $Q \subseteq P$ is a nonzero prime ideal, then $Q \in \mathcal{S}^{\perp}$. A similar assertion holds for $sp(\mathcal{S})$.

Proof. If $0 \neq d \in P$ and $dD = (AB)_t$ with $A \in \mathcal{S}$ and $B \in \mathcal{S}^{\perp}$, then $P \supseteq A$ or $P \supseteq B$. So $P \in sp(\mathcal{S})$ or $P \in \mathcal{S}^{\perp}$, but not both because $P_t \neq D$. For the second part, we may assume that Q is a prime t-ideal, so $Q \in \mathcal{S}^{\perp}$, by the first part. ¤

Lemma 11. Let $\mathcal{S}$ be a t-splitting set of ideals of D. Then
(a) $(AD_{\mathcal{S}})_t = D_{\mathcal{S}}$ for each $A \in sp(\mathcal{S})$, and
(b) $I = ((I \cap D)D_{\mathcal{S}})_t = (I \cap D)_{\mathcal{S}}$ for each t-ideal I of $D_{\mathcal{S}}$.

Proof. $\mathcal{S}$ is v-finite cf. Proposition 5, so we may apply [8, Proposition 1.8] and part (iv) of [8, Proposition 1.5] to finish the proof. ¤

Denote by $T(D)$ the ordered monoid of fractional t-ideals of D with the t-product and ordered by reverse inclusion and denote by $T_+(D)$ its positive cone, that is, $T_+(D) = \{A \in T(D) |\ A \subseteq D\}$. When $\mathcal{S}$ is a multiplicative set of ideals of D, $T(D_{\mathcal{S}}) \times_c T(D_{\mathcal{S}^\perp})$ stands for the cardinal product of the monoids $T(D_{\mathcal{S}})$ and $T(D_{\mathcal{S}^\perp})$. Our next result is an extension of [3, Theorem 4.12].

Theorem 12. If $\mathcal{S}$ is a t-splitting set of ideals of D, the map $\alpha : T(D) \to T(D_{\mathcal{S}}) \times_c T(D_{\mathcal{S}^\perp})$, $\alpha(I) = ((ID_{\mathcal{S}})_t, (ID_{\mathcal{S}^\perp})_t)$ is a monoid order-isomorphism.

Proof. Clearly, α is an order-preserving monoid homomorphism. It suffices to show that $\gamma = \alpha\ |_{T_+(D)}$: $T_+(D) \to T_+(D_{\mathcal{S}}) \times T_+(D_{\mathcal{S}^\perp})$ is a monoid order-isomorphism. Consider the map $\beta : T_+(D_{\mathcal{S}}) \times_c T_+(D_{\mathcal{S}^\perp}) \to T_+(D)$, $\beta(I, J) = ((I \cap D)(J \cap D))_t$ (note that $I \cap D \in \mathcal{S}^\perp$ and $J \cap D \in sp(\mathcal{S})$, cf. Remark 6). We prove that γ and β are inverse to each other. Indeed, if $A \in T_+(D)$, then $\beta(\gamma(A)) = ((AD_{\mathcal{S}})_t \cap D)((AD_{\mathcal{S}^\perp})_t \cap D)_t = A$ cf. Proposition 8. Conversely, let $(I, J) \in T_+(D_{\mathcal{S}}) \times_c T_+(D_{\mathcal{S}^\perp})$ and set $A = \beta(I, J) = ((I \cap D)(J \cap D))_t$. Since $J \cap D \in sp(\mathcal{S})$, $((J \cap D)D_{\mathcal{S}})_t = D_{\mathcal{S}}$, cf. Lemma 11. Again by Lemma 11, $((I \cap D)D_{\mathcal{S}})_t = I$. So $(AD_{\mathcal{S}})_t = ((I \cap D)D_{\mathcal{S}})_t = I$. Similarly, $(AD_{\mathcal{S}^\perp})_t = J$. Thus $\gamma(\beta(I, J)) = (I, J)$. ¤

The next result extends [3, Remark 4.13]. Denote by $TI(D)$ the group of fractional t-invertible t-ideals of D with the t-product and by $Cl_t(D)$ the t-class group of D, that is, the factor group of $TI(D)$ modulo its subgroup of principal fractional ideals. For $I \in TI(D)$, let $[I]$ denote the image of I in $Cl_t(D)$.

Remark 13. Let $\mathcal{S}$ be a t-splitting set of ideals of D. By Theorem 12, the map α given there induces an isomorphism $TI(D) \to TI(D_{\mathcal{S}}) \times TI(D_{\mathcal{S}^\perp})$. Moreover, if A is a principal fractional ideal of D, then both components of $\alpha(A)$ are principal. Consequently, α induces a surjective group homomorphism $\bar{\alpha} : Cl_t(D) \to Cl_t(D_{\mathcal{S}}) \times Cl_t(D_{\mathcal{S}^\perp})$, $\bar{\alpha}([I]) = ([(ID_{\mathcal{S}})_t], [(ID_{\mathcal{S}^\perp})_t])$. As documented in [3, Remark 4.13], $\bar{\alpha}$ need not be a monomorphism.

For a domain D, let t-Spec(D) (resp., t-Max(D)) denote the set of all t-prime ideals (resp., maximal t-ideals) of D.

Remark 14. Let $\mathcal{S}$ be a t-splitting set of ideals of D. From the proof of Theorem 12, we get a one-to-one correspondence between $\mathcal{S}^\perp \cap T_+(D)$ and $T_+(D_{\mathcal{S}})$ given by $A \mapsto (AD_{\mathcal{S}})_t$ and $I \mapsto I \cap D$. Restricting, we get a one-to-one correspondence between $\mathcal{S}^\perp \cap t$-Spec(D) and t-Spec$(D_{\mathcal{S}})$. By [4, Theorem 1.1], if $Q \in t$-Spec$(D_{\mathcal{S}})$, then $(D_{\mathcal{S}})_Q = D_{Q \cap D}$. Also, we get a one-to-one correspondence between $sp(\mathcal{S}) \cap t$-Spec(D) and t-Spec$(D_{\mathcal{S}^\perp})$. Note that by Proposition 10, the sets $sp(\mathcal{S}) \cap t$-Spec(D) and $\mathcal{S}^\perp \cap t$-Spec(D) give a partition of t-Spec(D). Similar correspondences hold when replacing t-Spec by t-Max.

Therefore, by Remark 14 and [4, Theorem 1.1], t-Max$(D_{\mathcal{S}^\perp}) = \{P_{\mathcal{S}^\perp};\ P \in sp(\mathcal{S}) \cap t$-Max$(D)\}$ and $(D_{\mathcal{S}^\perp})_{P_{\mathcal{S}^\perp}} = D_P$ for each $P \in sp(\mathcal{S}) \cap t$-Max$(D)$. Similarly,

t-Max$(D_{\mathcal{S}}) = \{P_{\mathcal{S}};\ P \in \mathcal{S}^{\perp} \cap t$-Max$(D)\}$ and $(D_{\mathcal{S}})_{P_{\mathcal{S}}} = D_P$ for each $P \in \mathcal{S}^{\perp} \cap$ t-Max(D).

Corollary 15. Let $\mathcal{S}$ be a t-splitting set of ideals of D. Then $D_{\mathcal{S}} = \cap\{D_P |\ P \in$ t-Max$(D) \cap \mathcal{S}^{\perp}\}$ and $D_{\mathcal{S}^{\perp}} = \cap\{D_P |\ P \in t$-Max$(D) \cap sp(\mathcal{S})\}$.

Proof. By the preceding paragraph, $D_{\mathcal{S}^{\perp}} = \cap\{(D_{\mathcal{S}^{\perp}})_Q |\ Q \in t$-Max$(D_{\mathcal{S}^{\perp}})\} =$ $\cap\{D_P |\ P \in t$-Max$(D) \cap sp(\mathcal{S})\}$. The other equality can be proved similarly. ¤

Let us recall from [10] that D is a PVMD if and only if D_P is a valuation domain for each maximal t-ideal P of D.

Proposition 16. Let $\mathcal{S}$ be a t-splitting set of ideals of D. Then every finite type t-ideal in $sp(\mathcal{S})$ is t-invertible if and only if $D_{\mathcal{S}^{\perp}}$ is a PVMD.

Proof. $(\Rightarrow)$ Let $Q \in t$-Max$(D_{\mathcal{S}^{\perp}})$ and $P = Q \cap D$. Then $P \in t$-Max$(D) \cap sp(\mathcal{S})$ by Lemmas 4 and 11.

Let J be a nonzero finitely generated ideal of D_P. Then $J = ID_P$ where I is a finitely generated ideal of D. Then $I_t = (AB)_t$ for some $A \in sp(\mathcal{S})$ and $B \in \mathcal{S}^{\perp}$. Since $P \in sp(\mathcal{S})$, $B * P$, and so $(ID_P)_t = (I_tD_P)_t = ((AB)_tD_P)_t = ((AB)D_P)_t =$ $(AD_P)_t$. Also, since I is finitely generated, I_t, and hence A_t is of finite type; so A_t is t-invertible. Note that P is a prime t-ideal of D; so $AA^{-1} * P$. Hence AD_P and ID_P are invertible, and thus ID_P is principal. So D_P is a valuation domain. Thus as $D_P \subseteq (D_{\mathcal{S}^{\perp}})_Q$, $(D_{\mathcal{S}^{\perp}})_Q$ is a valuation domain, and thus $D_{\mathcal{S}^{\perp}}$ is a PVMD.

$(\Leftarrow)$ Let $I \in sp(\mathcal{S})$ be a finite type t-ideal of D, and let $P \in t$-Max(D). If $P \notin sp(\mathcal{S})$, then $I * P$, and hence $ID_P = D_P$. Assume that $P \in sp(\mathcal{S})$. Then $P_{\mathcal{S}^{\perp}}$ is a t-ideal of $D_{\mathcal{S}^{\perp}}$ and $D_P = (D_{\mathcal{S}^{\perp}})_{P_{\mathcal{S}^{\perp}}}$. Since $D_{\mathcal{S}^{\perp}}$ is a PVMD, D_P is a valuation domain. Also, since I is a finite type t-ideal, ID_P is principal. Hence I is t-locally principal, and thus I is t-invertible, cf. [14, Proposition 2.6]. ¤

Our next result is a variant of [6, Theorem 2.2].

Proposition 17. Let Γ be a collection of t-invertible prime t-ideals of D and $\mathcal{S}$ the multiplicative set generated by Γ. Then the following statements are equivalent.

(a) $\mathcal{S}$ is a t-splitting set.

(b) $\cap_n P_1 \cdots P_n = 0$ for each sequence (P_n) of elements of Γ.

(c) $D_{\mathcal{S}^{\perp}}$ is a Krull domain.

Proof. Clearly, $\mathcal{S}^{\perp}$ is the set of ideals I of D contained in no $P \in \Gamma$. Note that $\Gamma \subseteq t$-Max(D) cf. [13, Proposition 1.3].

$(a) \Rightarrow (c)$ Let $Q \in t$-Max$(D) \cap sp(\mathcal{S})$ and $Q' \subseteq Q$ a minimal prime of a principal ideal. Then Q' is a t-ideal and $Q' \in sp(\mathcal{S})$ cf. Proposition 10. Then $Q' \supseteq P_1 \cdots P_n$ for some $P_i \in \Gamma$. Hence $Q' = P_i = Q$ because $P_i \in t$-Max(D). Thus t-Max$(D) \cap$ $sp(\mathcal{S}) = \Gamma$ and each $P \in \Gamma$ has height one. By Lemma 4, $P_{\mathcal{S}^{\perp}}$ is t-invertible in $D_{\mathcal{S}^{\perp}}$ for each $P \in \Gamma$. By the paragraph after Remark 14, t-Max$(D_{\mathcal{S}^{\perp}}) = \{P_{\mathcal{S}^{\perp}} |\ P \in \Gamma\}$ and each $P_{\mathcal{S}^{\perp}}$ has height one, because $(D_{\mathcal{S}^{\perp}})_{P_{\mathcal{S}^{\perp}}} = D_P$. By [15, Theorem 3.6], $D_{\mathcal{S}^{\perp}}$ is a Krull domain.

$(c) \Rightarrow (b)$ Let (P_n) be a sequence of elements of Γ and $P = P_n$ for some n. Clearly $P \notin \mathcal{S}^{\perp}$. As P is t-invertible, we have $(PD_{\mathcal{S}^{\perp}})_t = P_{\mathcal{S}^{\perp}}$ (see the proof of Lemma 4), so $P_{\mathcal{S}^{\perp}}$ is a prime t-ideal of $D_{\mathcal{S}^{\perp}}$. Since $D_{\mathcal{S}^{\perp}}$ is a Krull domain, we get $\cap_n P_1 \cdots P_n \subseteq \cap_n (P_1)_{\mathcal{S}^{\perp}} \cdots (P_n)_{\mathcal{S}^{\perp}} = 0$.

$(b) \Rightarrow (a)$ Assume that $\cap_n P_1 \cdots P_n = 0$ for each sequence (P_n) of ideals of Γ. Let $0 \neq d \in D$. Since each $P \in \Gamma$ is t-invertible, if I is a nonzero ideal contained in P, we get $I_t = (PJ)_t$ with $J = P^{-1}I$. We use repeatedly this factorization property starting with $I = dD$. By our assumption on Γ, we get $dD = (P_1 \cdots P_n J)_t$ for some $P_1, ..., P_n \in \Gamma$, $n \geq 0$ and some ideal J contained in no $P \in \Gamma$, thus $J \in \mathcal{S}^{\perp}$. ¤

We recall that a Mori domain is a domain satisfying the ascending chain condition on integral divisorial ideals.

Corollary 18. A collection of t-invertible prime t-ideals of a Mori domain generates a t-splitting set.

Corollary 19. A collection of t-invertible uppers to zero in $D[X]$ generates a t-splitting set.

Recall that with the realization of the power of splitting sets came various extensions of Nagata's theorem for UFD's (see e.g. [2]). Now the question is what can the t-splitting sets of ideals do for us? In fact they can deliver a somewhat modified version of Nagata type Theorems.

An integral domain D is said to be of finite t-character if every nonzero nonunit of D belongs to only finitely many maximal t-ideals of D.

Proposition 20. Let $\mathcal{S}$ be a t-splitting set of ideals of an integral domain D, and suppose that every proper ideal in $\mathcal{S}$ is contained in at most a finite number of maximal t-ideals of D. Then $D_{\mathcal{S}}$ is a ring of finite t-character if and only if D is a ring of finite t-character.

Proof. By Proposition 10 and the paragraph preceding Corollary 15, if P is a maximal t-ideal of D, then either $P \in sp(\mathcal{S})$ or $P \in \mathcal{S}^{\perp}$ and that $t\text{-Max}(D_{\mathcal{S}}) = \{P_{\mathcal{S}} | P \in \mathcal{S}^{\perp} \cap t\text{-Max}(D)\}$. For $0 \neq d \in D$, let $dD = (AB)_t$, where $A \in sp(\mathcal{S})$ and $B \in \mathcal{S}^{\perp}$. Since $A \in \mathcal{S}$, there are only a finite number of maximal t-ideals in $sp(\mathcal{S})$ containing A (and hence d). Moreover, since $t\text{-Max}(D_{\mathcal{S}}) = \{P_{\mathcal{S}} | P \in \mathcal{S}^{\perp} \cap t\text{-Max}(D)\}$, the number of maximal t-ideals in $\mathcal{S}^{\perp}$ containing d is finite. Therefore, D is of t-finite character. The converse is straightforward from the above observation. ¤

This result can be put to direct use in a number of situations. In the following, we address a few of them.

Corollary 21. Let D be an integral domain and let $\mathcal{S}$ be a t-splitting set of ideals of D generated by height-one prime ideals. Suppose that every proper ideal in $\mathcal{S}$ is contained in at most a finite number of maximal t-ideals of D. Then $D_{\mathcal{S}}$ is a ring of finite t-character if and only if D is a ring of finite t-character.

An integral domain D is called a weakly Krull domain if $D = \cap_{P \in X^1(D)} D_P$ and this intersection has finite character. According to [11], a ring of Krull type is an integral domain which is a locally finite intersection of essential valuation overrings. The ring D of Krull type is an independent ring of Krull type if each prime t-ideal of D lies in a unique maximal t-ideal and a generalized Krull domain if D is weakly Krull.

Corollary 22. Let $\mathcal{F}$ be a family of height-one t-invertible prime t-ideals of an integral domain D. Let $\mathcal{S}$ be a multiplicative set of ideals generated by $\mathcal{F}$ and suppose that every nonzero nonunit of D belongs to at most a finite number members of $\mathcal{F}$.

(1) D is a weakly Krull domain if and only if $D_{\mathcal{S}}$ is.
(2) D is a generalized Krull domain if and only if $D_{\mathcal{S}}$ is.
(3) D is a ring of Krull type if and only if $D_{\mathcal{S}}$ is.

(4) D is an independent ring of Krull type if and only if $D_{\mathcal{S}}$ is.
(5) D is a PVMD if and only if $D_{\mathcal{S}}$ is.

Proof. The proof consists in noting that every t-invertible prime t-ideal P is a maximal t-ideal [13, Proposition 1.3] and that P being of height-one implies that D_P is a discrete valuation domain. The rest depends upon recalling the definitions of the respective notions. ¤

In this vein it would be interesting to record the following result.

Corollary 23. Let X be an indeterminate over the integral domain D and $S = \{f \in D[X] | A_f^{-1} = D\}$. Then D is a ring of Krull type if and only if $(D[X])_S$ is a Bezout domain of finite character.

Proof. Recall that D is a PVMD if and only if $D[X]_S$ is a Bezout domain [14, Theorem 3.7] and that D is of finite character if and only if $D[X]$ is [9, Exercise 1, pp.537]. So the result follows from Corollary 22(4) because the set $\mathcal{S} := \{I \subseteq D[X] | I$ is an ideal of $D[X]$ such that $f \in I$ for some $f \in S\}$ is a t-splitting set of ideals. ¤

Just to give an idea of how these results can be extended we state the following. Let $*$ be a star operation on an integral domain D, and let $*_s$ be the finite type star operation induced by $*$, i.e., $I^{*_s} = \cup\{F^* | F \subseteq I$ is finitely generated$\}$ for any $I \in F(D)$. Then D is called a Prüfer $*$-multiplication domain if every finitely generated ideal of D is $*_s$-invertible. It is clear that Prüfer $*$-multiplication domains are PVMDs because $I^{*_s} \subseteq I_t$.

Proposition 24. Let D be a domain, $*$ a star operation of finite type on D, $\mathcal{F}$ a family of maximal height-one principal primes of D and S the multiplicative set generated by $\mathcal{F}$. Suppose that each nonzero nonunit of D is contained in at most a finite number of members of $\mathcal{F}$. Then D is of $*$-finite character (resp., a Prüfer $*$-multiplication domain) if and only if D_S is of $*$-finite character (resp., a Prüfer $*$-multiplication domain).

We note that if the finite character star operation $*$ is the identity star operation d that takes $A \mapsto A$ for all $A \in F(D)$, then a Prüfer $*$-multiplication domain is a Prüfer domain. Thus for $* = d$ Proposition 24 gives us the following corollary.

Corollary 25. Let D be domain, $\mathcal{F}$ a family of height-one principal primes that are also maximal ideals and S the multiplicative set generated by $\mathcal{F}$. Suppose that every nonzero nonunit of D belongs to at most a finite number of members of $\mathcal{F}$. Then D is a Pr¨ufer domain of finite character if and only if D_S is a Pr¨ufer domain of finite character.

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