QUESTION (HD 0602): Kaplansky, in his book on Commutative Rings, calls an integral domain D an S-domain if for every height one prime P of D we have height $(P[X]) = 1$ where X is an indeterminate over D. He then moves on to define strong S-rings, to show that if R is a Noetherian ring then dim $(R[X]) =$ dim $(R) + 1$. Are there any examples of S-domains, or were they introduced to flash Seidenberg's name?

ANSWER. It appears that some introduction, for a general reader, to the terms you have thrown at us, is in order. A prime ideal P of a ring R is said to be of height n if there is a chain of prime ideals $P = P_0 \supset P_1 \supset P_2 \supset ... \supset P_n$ of length n such that any other chain of prime ideals properly descending from P has length less than or equal to n. A prime ideal is said to have infinite height (or rank) if for each n there is a chain of prime ideals of length greater than n descending from P. Thus a minimal prime ideal of a ring has height 0. A ring R is said to have Krull dimension $n < \infty$ if R contains at least one prime ideal of height n and if every other prime ideal of R is of height less than or equal to n. (In this case we say that R has finite Krull dimension or simply that "R is finite dimensional".) Now let R be n dimensional and let X be an indeterminate over R and it was natural to ask: What is the Krull dimension of $R[X]$? A. Seidenberg [S1](Pacific J. Math. 3(1953) 505-512) and [S2](Pacific J. Math. 4(1954) 603-614) showed, using a set of ingenious results and some work of Krull, that the Krull dimension of R can be any integer from $n + 1$ to $2n + 1$.

Kaplansky on his part, to simplify matters a great deal, introduced the notion of an S-domain as an auxiliary notion apparently to introduce the notion of a strong S-ring as a ring R such that for every prime ideal P the integral domain R/P is an S-domain. Using the fact that if P is a height one prime ideal of an S-domain D then height($P[X]$) $= 1$. He shows, via his Theorem 39, that if R is an n-dimensional strong S-ring and X an indeterminate over R then dim($R[X]$) $= n + 1$. He mentions that a Noetherian ring is a strong S-ring and later, via Theorem 68, he establishes that a valuation ring is a strong S-ring. This should provide you with plenty of examples of S-domains. But there are plenty more. For instance it was shown in Fontana and Kabbaj's [FK](J. Pure Appl. Algebra, 63(3)(1990) 231-245.) and in Anderson, Anderson and Zafrullah's [AAZ](Houston J. Math. 17(1)(1991) 109-129.), that $D[X]$ is an S-domain for any integral domain D. What is more important than this result is the following result (Lemma 3.1) from [AAZ].

Proposition. The following are equivalent for an integral domain D.

1. D is an S-domain.
2. D_P is an S-domain for each height one prime ideal P.
3. For each height one prime P of D the integral closure of D_P is a Prüfer domain.

(Recall that D is a Prüfer domain if every finitely generated ideal of D is invertible and that D is Prüfer if and only if D_M is a valuation domain for each maximal ideal M.)

This result which is directly related to Seidenberg's work opens the floodgate of examples and non-examples.

(i). If D has no height one prime then D is vacuously an S-domain.

(ii). Refer to HD0311 for a brief introduction to the v-ideals and t-ideals. As established in HD0302 every integral t-ideal is contained in a maximal t-ideal, which is necessarily prime. Let X be an indeterminate over D. Call a prime ideal Q of $D[X]$ an upper to zero if $Q \neq (0)$ and $Q \cap D = (0)$. An upper to zero being of height one is a minimal prime of a principal ideal and hence a prime t-ideal (cf HD0302). Call D a UMT domain if every upper to zero of $D[X]$ is a maximal t-ideal. This notion was introduced in Houston and Zafrullah's [HZ](Comm. Algebra 17(1989), 1955-1969) and further studied by Fontana, Gabelli and Houston in [FGH](Comm. Algebra 26(4)(1998) 1017-1039). Recall from [FGH, Theorem 1.5] that D is a UMT domain if and only if for every prime t-ideal P the integral closure of D_P is a Prüfer domain. But since every height one prime P is a t-ideal we conclude that every UMT domain is an S-domain. Now it so happens that a lot of integral domains of interest are indeed S-domains. A result in [HZ] (Proposition 3.2) says that an integral domain D is a PVMD if and only if D is an integrally closed UMT domain. (Recall that D is a PVMD (Prüfer v-multiplication domain) if for every finitely generated ideal A of D we have $(AA^{-1})_t = D$ and note that every GCD domain is a PVMD.)

(iii). It may be noted that while every UMT domain is an S-domain, not every S-domain is a UMT domain. As it was mentioned above, every Noetherian domain is an S-domain. It was shown in Theorem 3.7 of [HZ] that a Noetherian domain D is a UMT domain if and only if every prime t-ideal of D is of height one. Now there do exist Noetherian domains in which maximal t-ideals (maximal primes of principal ideals) are of height greater than one see Example 4.1 of Houston and Zafrullah [HZTV](Michigan Math. J. 35(1988), 291-300.) To see another example of an S-domain recall that a prime ideal P that is minimal over a proper nonzero ideal of the form $(a) : (b) = \{r \in D : rb \in (a)\}$ is called an associated prime of a principal ideal by Brewer and Heinzer [BH](Duke Math J. 41(1974) 1-7). Let us call such a prime an associated prime (of D). It was shown in [BH] that $D = \cap D_P$ where P ranges over associated primes of D. Call D a P-domain if D_P is a valuation domain for every associated prime. Now, being an intersection of valuation overrings a P-domain is integrally closed and because of the definition D_P is a valuation domain for every height one prime we conclude that a P-domain is an S-domain. Now there are P-domains that are not PVMD's and hence not UMT domains. The P-domains were introduced by Mott and Zafrullah in [MZ](Manuscripta Math. 35(1-2)(1981) 1-26), further studied by Papick in [P](Pacific J. Math. 105, no. 1 (1983), 217–226) and mentioned by Fontana and Kabbaj in [FK2](Proc. Amer. Math. Soc. 132 (2004), no. 9, 2529–2535 (electronic)). For examples of P-domains that are not PVMD's look up [MZ] and Zafrullah [Z](Comm. Algebra 31(5)(2003) 2497-2540) and references there. Also check out HD0313.

(iv). What is surprising is that the first example of a non-S-domain was also provided by Seidenberg as it was crucial for what he aimed to prove. From the above Proposition, to show that D is not an S-domain we need to provide at least one height one prime P such that the integral closure of D_P is not a Prüfer domain. We give an example of a non-S-domain that will answer many questions. The example is that of a one dimensional, completely integrally closed quasi local domain D that is not a valuation domain. Being completely integrally closed D is integrally closed and not being a valuation domain D cannot have Prüfer integral closure. So, D cannot be an S-domain. The bad part of this example is that it is in two papers by Nagata [N1](NagoyaMath. J. 4(1952) 29-33) and [N2](Nagoya Math. J. 9(1955) 209-212.), where the second paper is a corrigendum. But the example is good and a sort of must read. Now let us reap the benefits of this example. This example tells us that a completely integrally closed integral domain does not have to be an

S-domain. Now recall that an integral domain D is a v-domain if for every finitely generated ideal A of D we have $(AA^{-1})_v = D.$ (Compare this definition with that of a PVMD.) Now as shown in Gilmer's book on multiplicative ideal theory [G](Marcel-Dekker, 1972.) an integral domain is completely integrally closed if and only if for each nonzero ideal A of D we have $(AA^{-1})_v = D$ ([G], Theorem 34.3). So a completely integrally closed domain is a v-domain. The Nagata example then is a v-domain that is not an S-domain. For examples of S-domains arising from various polynomial ring constructions the reader may consult [AAZ] and [FK].

(v). Marco Fontana, on reading an earlier version, suggested that I should also include simpler examples of non S domains. Indeed any one dimensional quasi local domain whose integral closure is not a Prüfer domain is an example of a non S-domain. Here are some: (a) $D = \mathbf{Q} + X\mathbf{R}[[X]] = \{f \in \mathbf{R}[[X]] : f(0) \in \mathbf{Q}\}$, i.e., the set of all power series with real coefficients whose constant terms are rational. It is easy to check that the inregral closure D' of D is $\widetilde{\mathbf{Q}} + X\mathbf{R}[[X]],$ which is not a valuation domain. Here $\widetilde{\mathbf{Q}}$ is the algebraic closure of $\mathbf{Q}$ in $\mathbf{R}$. (Here $\mathbf{Q}$ and $\mathbf{R}$ represent, respectively, the set of rational numbers and real numbers.) Indeed with a little bit of knowledge of localization one can also show that $D = \mathbf{Q} + X\mathbf{R}[X],$ that is the set of polynomials over $\mathbf{R}$ with constant terms in $\mathbf{Q}$ is also an example of a one dimensional integral domain that is not an S-domain. This too is a consequence of the fact that if $D = \mathbf{Q} + X\mathbf{R}[X]$ then $D' = \widetilde{\mathbf{Q}} + X\mathbf{R}[X].$

The late Professor Irving Kaplansky, the author of "Commutative Rings", was a man of vision who contributed a great deal to Mathematics. In addition he knew the art of conveying great ideas briefly and simply. The notions of S-domains and strong S-rings are examples of Kaplansky introducing a notion that on inspection turned out to be an interesting topic on its own. Before I get down to giving examples and references, one word of caution. Respect your elders. S-domains were named after Seidenberg not because he was a member of a cult or some organization, but because he did something significant using a notion close to S-domains. He provided examples of domains D to show that if the (Krull) dim(D) is n and X an indeterminate then $\dim(D[X])$ can be any number from $n+1$ to $2n+1,$ as mentioned earlier, and it is no mean feat. So the man did deserve to have some rings named after him.

Now Kaplansky's result that a valuation ring is a strong S-ring was further developed into: a Prüfer domain is a strong S-ring by Malik and Mott in [MM](J. Pure Appl. Algebra, 28(3)(1983)249-264) and this led to Kabbaj's work. For a brief overview of Kaplansky caused research on strong S-domains and S-domains look up my survey article [Z]. If you are interested you can chase the references given in [Z] to see the full scope of the notions.