QUESTION: (HD0702) Kronecker had associated, via "Kronecker function rings", a UFD with each ring of algebraic numbers years before Dedekind proved unique factorization of ideals of a ring of algebraic integers, of a special kind. Then why is it that we see Dedekind and Dedekind domains everywhere yet no mention of Kronecker?

ANSWER: (Preliminary answer, I plan to write a more detailed response later.)

You are right, Kronecker essentially wrote in 1859 the paper [L. Kronecker, Grundzüge einer arithmetischen der algebraischen Grössen, J. Reine Angew. Math., 92(1882), 1-122; Werke 2, 237-387 (K. Hensel, Editor, 5 volumes published from 1895-1930, Teubner Leipzig) reprint, Clesea 1968] that you refer to. But he did not use "Kronecker function rings" as we know them today. He used the notion of divisors and of course he associated a PID with each ring of algebraic integers, however he was essetially interested in getting the GCD's. Also as you can see Kronecker's paper was published in 1882, a decade after Dedekind's work [R. Dedekind, Supplement XI to Vorlesungen über Zahlentheorie von Dirichlet. Gesammelte mathematische Werke, (Fricke, Noether and Ore Editors) Vol. 3, 1-222. Vieweg, 1930-1932] in which he established the result on the unique factorization of ideals that you mention. Dedekind's paper was clearly written and Kronecker had an obscure style. (Kronecker's work was "popularized" later by Krull who introduced the generalized notion of a Kronecker function ring, via star operations satisfying a specific property.) But Kronecker's obscurity of expression does not seem to be the reason for the ensuing popularity of Dedekind's work. Dedekind used notions such as Kummer's ideal numbers to introduce ideals that kept the study within the confines of the domains of interest to him. Consequently the results that he proved developed, later, into results on the unique factorization of non-zero ideals of Dedekind domains as products of powers of maximal ideals. The Dedekind domains had such rich structure that results proved for them heralded new concepts and approaches in ring theory. The idea of the ideal class group is one of the aspects of Dedekind domains R that consists of the semigroup of nonzero fractional ideals of R modulo the group of nonzero principal ideals of R. The ideal class semigroup of R may be denoted by $Cl(R)$. For a Dedekind domain R, $Cl(R)$ is a group because in a Dedekind domain every nonzero fractional ideal is invertible. Now there are Dedekind domains whose class group is torsion. It is easy to see from the definition of a class group that a Dedekind domain R is a PID if and only if $Cl(R) = 0$.

Krull [K, Beiträge zur Arithmetik Kommutativer Integritätsbereiche, I - II, Math. Z. 41(1936)545-577, 665-679] introduced the general form of Kronecker function ring and the notion of a Krull domain, using which we can assciate a PID to each Dedekind domain which can be used to show that the nonzero ideals of a Dedekind domain have a sort of unique factorization. Later we shall show how it works, but the point is, this Kronecker method is a machine to associate a PID with a Dedekind domain. It associates a PID with a PID, and a PID with a Dedekind domain with non-torsion class group and a PID with a Dedekind domain with torsion class group. That may be the reason why

Kronecker's work >in this area< does not seem to be very prominent.

Finally the last part of your question shows that you do not know much about Kronecker and Dedekind. I learned about Kronecker's delta way before I studied Dedekind cuts. For me, both are my Mathematical forefathers and so I respect them both. But come to think of it, I see Kronecker everywhere, at least as much as Dedekind. In algebraic geometry I see Weyl divisors which are essentially based on Kronecker's divisors. Then in field theory Kronecker seems to be at work with a result that if f is an irreducible polynomial over a field F then there is an extension E (of F) that contains a root of f. There are many more things that the man did which have a direct bearing on how and what Mathematics we do today, of course same about Dedekind.

Consultation with Marco Fontana has been a great help and so has been his paper with Alan Loper, "An historical overview of Kronecker function rings", Nagata rings and related star and semistar operations, in " Multiplicative Ideal Theory in Commutative Algebra: A tribute to the work of Robert Gilmer", Jim Brewer, Sarah Glaz, William Heinzer, and Bruce Olberding Editors, Springer 2006.

In a more detailed response, that will replace this one, I plan to show how to associate a PID with a Dedekind domain using basic ideas and I plan to discuss Kronecher's divisors approach.

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