QUESTION: (HD0802) I am interested in learning about the generalizations of Prüfer domains called v-domains and Prüfer v-multiplication domains, but they are studied using the star operations, which I am not very familiar with. Is there a way of defining these concepts without any mention of star operations?

ANSWER: You have raised an interesting and important question. Often other Mathematicians are not attracted to the notions of v-domain and its specializations because the jargon of star operations appears forbidding, though it is not. In the following I provide characterizations, of some of these domains. In my answer I prove statements, that, when used as definitions, do not mention any star operations. Note that "officially" an integral domain is a v-domain (resp., a Prüfer v-multiplication domain (PVMD)) if for every nonzero finitely generated (fractional) ideal A of D we have $(AA^{-1})_v = D$ (resp., $(AA^{-1})_t = D$). So to understand the justification of the new definitions you will need some basic knowledge of the star operations. For this the best source available is sections 32 and 34 of Gilmer [2]. For now, if you are in a hurry look up HD0311 for star operations and note the following. The letter D denotes an integral domain with quotient field K. By $F(D)$ we denote the set of all nonzero fractional ideals of D and by $f(D)$ the members of $F(D)$ that are finitely generated. We use, for D-submodules A, B of K (such as the members of $F(D)$), the notation $A : B$ to denote the set $\{x \in K : xB \subseteq A\}$ and denote $D : A$ by A^{-1} which belongs to $F(D)$ whenever A does. If $A \subseteq B$ then $A^{-1} \supseteq B^{-1}$, for $A, B \in F(D)$. Moreover from the definition it follows that $(AA^{-1}) \subseteq D$ and $D^{-1} = D$. Note that for $A \in F(D)$, $A_v = (A^{-1})^{-1} = D : (D : A)$ and that $A_t = \cup\{F_v : F \subseteq A \wedge F \in f(D)\}$. It can be shown easily that $(A_v)^{-1} = A^{-1}$. If $A \in F(D)$ is such that $A = A_v$ (resp., $A = A_t$) we say that A is a v-ideal (resp., a t-ideal). A v-ideal is also called a divisorial ideal. A^{-1} is a v-ideal, and every invertible ideal is a v-ideal and a t-ideal. Also if there is a finitely generated fractional ideal B such that $A = B_v$ we say that A is a v-ideal of finite type. For a star operation $*$ call $A \in F(D)$ $*$-invertible if there is $B \in F(D)$ such that $(AB)^* = D$. It can be shown that in this case $B^* = A^{-1}$. (You can take $*$, here, as a general name for the v- and t-operations and note that an invertible ideal is both v- and t-invertible.) So, D is a v-domain (resp., PVMD) if every $A \in f(D)$ is v-invertible (resp., t-invertible), as indicated earlier. It can be shown that $A \in f(D)$ is t-invertible if and only if A is v-invertible and A^{-1} is a v-ideal of finite type.

Lemma 1. An ideal $A \in F(D)$ is v-invertible if and only if $A^{-1} : A^{-1} = D$.

Proof. Suppose that $A^{-1} : A^{-1} = D$. Let $x \in (AA^{-1})^{-1} \supseteq D$. Then $x(AA^{-1}) \subseteq D$ or $xA^{-1} \subseteq A^{-1}$ or $x \in A^{-1} : A^{-1} = D$. So $(AA^{-1})^{-1} \subseteq D$ and we have $(AA^{-1})^{-1} = D$. This gives $(AA^{-1})_v = D$.

Conversely if A is v-invertible then $(AA^{-1})^{-1} = D$. Let $x \in A^{-1} : A^{-1} \supseteq D$. Then $xA^{-1} \subseteq A^{-1}$. Multiplying both sides by A and applying the v-operation we get $x \in D$. So, $D \subseteq A^{-1} : A^{-1} \subseteq D$ and the equality follows.

Proposition 2. For an integral domain D the following are equivalent.

(0) D is a v-domain.

(1) $A^{-1} : A^{-1} = D$ for each $A \in f(D)$.

(2) $A_v : A_v = D$ for each $A \in f(D)$.

(3) $A^{-1} : A^{-1} = D$ for each two generated $A \in f(D)$.

(4) $((a) \cap (b)) : ((a) \cap (b)) = D$ for all $a, b \in D\backslash\{0\}$.

Proof. (0) $\Leftrightarrow$ (1) follows from Lemma 1 once we note that D is a v-domain if and only if each $A \in f(D)$ is v-invertible.

(0) $\Rightarrow$ (2). Let $x \in A_v : A_v \supseteq D$. Then $xA_v \subseteq A_v$. Multiplying both sides by A^{-1} and applying the v-operation we get $x(A_vA^{-1})_v \subseteq (A_vA^{-1})_v$. But by (0) $(A_vA^{-1})_v = D$ and so $x \in D$. This forces $D \subseteq A_v : A_v \subseteq D$.

(2) $\Rightarrow$ (0). Let $x \in (A_vA^{-1})^{-1} \supseteq D$. Then $x(A_vA^{-1}) \subseteq D$. But then $xA_v \subseteq A_v$ which gives $x \in A_v : A_v = D$. But then $D \subseteq (A_vA^{-1})^{-1} \subseteq D$, which means that every $A \in f(D)$ is v-invertible.

(1) $\Rightarrow$ (3) is obvious.

(3) $\Rightarrow$(4). Let $a, b \in D$ and by (3) $(a,b)^{-1} : (a,b)^{-1} = D$ or $\frac{(a)\cap(b)}{ab} : \frac{(a)\cap(b)}{ab} = D$ which is the same as $((a) \cap (b)) : ((a) \cap (b)) = D$.

(4) $\Rightarrow$ (0). Recall that D is a v-domain if and only if every two generated nonzero ideal of D is v-invertible [6]. Now let $x \in ((a,b)(a,b)^{-1})^{-1} \supseteq D$, where $a, b \in D\backslash\{0\}$. Then $x(a,b)(a,b)^{-1} \subseteq D$. Or $x(a,b)^{-1} \subseteq (a,b)^{-1}$. Or $x\frac{(a)\cap(b)}{ab} \subseteq \frac{(a)\cap(b)}{ab}$. Or $x((a)\cap(b)) \subseteq ((a)\cap(b))$ or $x \in ((a)\cap(b)) : ((a)\cap(b)) = D$. This forces $D \subseteq ((a,b)(a,b)^{-1})^{-1} \subseteq D$.

An immediate consequence of the above Proposition is the following characterization of PVMD's, which stems from the fact that D is a PVMD if and only if every two generated nonzero ideal of D is t-invertible [3] if and only if every two generated ideal (a,b) of D is v-invertible such that $(a,b)^{-1} = (x_1, x_2, ..., x_r)_v$ where $r \in N$. Note that $(a,b)^{-1} = (x_1, x_2, ..., x_r)_v$ implies that $\frac{(a)\cap(b)}{ab} = (x_1, x_2, ..., x_r)_v$. Multiplying both sides by ab and using the definition of the v-operation we have $(a) \cap (b) = (abx_1, abx_2, ..., abx_r)_v$. Recall from the introduction that a fractional v-ideal A is said to be a v-ideal of finite type if there exist $a_1, a_2...a_n \in A$ such that $A = (a_1, a_2...a_n)_v$. Call an integral domain D a v-finite conductor (v-FC-) domain if $(a) \cap (b)$ is a v-ideal of finite type, for every pair $a, b \in D\backslash\{0\}$.

You may say that in the above definition of v-FC-domains there is still a mention of the v-operation. We have a somewhat contrived solution for this, in the form of the following characterization of v-FC-domains.

Proposition 3. An integral domain D is a v-FC-domain if and only if for each pair a, b in $D\backslash\{0\}$ there exist $y_1, y_2, ..., y_n \in K\backslash\{0\}$ such that $(a,b)_v = \bigcap y_iD$. Consequently D is a v-FC-domain if and only if for each pair a, b in $D\backslash\{0\}$ there exist $z_1, z_2, ..., z_m \in K\backslash\{0\}$ such that $((a) \cap (b))^{-1} = \bigcap z_iD$.

Proof. Let D be a v-FC-domain and let $a, b \in D\backslash\{0\}$. Then there are $a_1, a_2...a_n$ such that $(a) \cap (b) = (a_1, a_2...a_n)_v$. Dividing both sides by ab we get $\frac{(a)\cap(b)}{ab} = (a_1/ab, a_2/ab...a_n/ab)_v$. But $\frac{(a)\cap(b)}{ab} = (a,b)^{-1}$. This gives $(a,b)_v = ((a_1/ab, a_2/ab...a_n/ab)_v)^{-1} = \bigcap \frac{ab}{a_i}D$. Conversely if $(a,b)_v = \bigcap y_iD$ then $\frac{(a)\cap(b)}{ab} = ((a,b)_v)^{-1} = (\bigcap y_iD)^{-1}$. This gives $\frac{(a)\cap(b)}{ab} = (\sum \frac{1}{y_i}D)_v$ by [4, Lemma 1.1]. Or $(a) \cap (b) = (\sum \frac{ab}{y_i}D)_v$. For the "consequently" part note that $((a) \cap (b))^{-1} =$

$\frac{1}{ab}(a,b)_v$.

Corollary 4. For an integral domain D the following are equivalent.

(1) D is a PVMD

(2) for all $a, b \in D\backslash(0)$ we have $((a) \cap (b))^{-1}$ a finite intersection of principal fractional ideals and $((a) \cap (b)) : ((a) \cap (b)) = D$

(3) D is a v-FC-domain and for all $a, b \in D\backslash(0)$ we have $((a) \cap (b)) : ((a) \cap (b)) = D$.

Recall that D is called a finite conductor (FC-) domain if $((a) \cap (b))$ is finitely generated for each pair $a, b \in D$. Just to show how far we have traveled since [9] we state and prove the following corollary.

Corollary 5. An integrally closed FC-domain is a PVMD.

Proof. We first note that since D is integrally closed $A : A = D$ for every finitely generated ideal A of D. So for each pair $a, b \in D\backslash\{0\}$, since D is FC, $((a) \cap (b)) : ((a) \cap (b)) = D$. But this makes D a v-domain, by Proposition 2 and a PVMD by Corollary 4.

While the result in Corollary 4 was the main result of [9] that paper contained many useful techniques. For example [9] was the first in the literature to introduce the formula $(AD_S)_v = (A_vD_S)_v$ where A is a finitely generated (nonzero) ideal of D and S is a multiplicative set of D, among other useful techniques.

Lemma 1 can also be instrumental in characterizing completely integrally closed (CIC-) domains. There is enough information for CIC-domains in section 34 of [2]. Also the approach in this answer leads to a characterization of Krull domains in a manner similar to the characterization of v-domains leading to the characterization of PVMD's.

Proposition 6. For an integral domain D the following are equivalent:

(1) D is a CIC-domain

(2) $A^{-1} : A^{-1} = D$ for all $A \in F(D)$

For the proof note that D is CIC if and only if every $A \in F(D)$ is v-invertible [2, Proposition 34.2, Theorem 34.3] and (2) provides precisely that by Lemma 1.

Proposition 7. The following are equivalent for an integral domain:

(1) D is a Krull domain;

(2) for each $A \in F(D)$ there exist $y_1, y_2, ..., y_n \in A$ such that $A^{-1} = \cap \frac{1}{y_i} D$ and for all $a, b \in D\backslash\{0\}$, $((a) \cap (b)) : ((a) \cap (b)) = D$;

(3) for each $A \in F(D)$ there exist $x, y \in A$ such that $A^{-1} = \frac{1}{x} D \cap \frac{1}{y} D$ and for all $a, b \in D\backslash\{0\}$, $((a) \cap (b)) : ((a) \cap (b)) = D$.

Before we prove Proposition 7 it seems pertinent to give some introduction. An integral domain D is called a Mori domain if D satisfies ACC on integral divisorial ideals. Different aspects of Mori domains were studied by Toshio Nishimura in a series of papers. For instance in [8] he showed that a domain D is a Krull domain if and only if D is a Mori domain and completely integrally closed. An integral domain D is a Mori domain if and only if for each $A \in F(D)$ there exist $y_1, y_2, ..., y_n \in A\backslash\{0\}$ such that $A_v = (y_1, y_2, ..., y_n)_v$ [7, Lemma 1], and it is easy to see that this characterization is equivalent to "for each $A \in F(D)$

there exist $y_1, y_2, ..., y_n \in A\backslash\{0\}$ such that $A^{-1} = \cap \frac{1}{y_i} D$". For a quick review of Krull domains the reader may consult the first few pages of [1]. A number of characterizations of Krull domains can be found in [4, Theorem 2.3]. The one that we can use here is: D is a Krull domain if and only if each $A \in F(D)$ is t-invertible. Which means that D is a Krull domain if and only if for each $A \in F(D)$, A is v-invertible and A^{-1} is of finite type. For another proof of a Mori domain being a Krull domain see [11, Corollary 2.2].

Proof. From the above discussion (1) $\Leftrightarrow$ (2). Next (3) $\Rightarrow$ (1) follows from (2)$\Leftrightarrow$ (1). For (1) $\Rightarrow$ (3) note that for all $a, b \in D\backslash\{0\}$, $((a)\cap(b)) : ((a)\cap(b)) = D$, because of (1) $\Leftrightarrow$(2). For the remaining part recall from [5, Proposition 1.3] that if D is a Krull domain then for every $A \in F(D)$ there exist $x, y \in A$ such that $A_v = (x, y)_v$.

Remark 8. In (2) of Proposition 7 we cannot say that for every $A \in F(D)$ the inverse A^{-1} is expressible as a finite intersection of principal fractional ideals, because this would be equivalent to A_v being of finite type for each $A \in F(D)$. But there do exist non-Mori domains D such that A_v is of finite type for all $A \in F(D)$. For a discussion of those examples you may consult [10, section 2].

Remark 9. I hope that you realize that while the above results provide definitions of those concepts in terms of mainstream algebra they also indicate the importance of star operations as a means of getting deeper than where the mainstream techniques could not help.

Remark 10. I am grateful to Said El Baghdadi for reading an earlier version of the answer and catching a really bad error. David Dobbs, Marco Fontana and Evan Houston also helped.

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