QUESTION: (HD0805) Let R be an integral domain which satisfies ACCP and I a non zero ideal of R. If R/I is an integral domain which is a homomorphic image of R. Does R/I also satisfy ACCP?

ANSWER: If R is any commutative ring and I any ideal of R the factor ring R/I is a homomorphic image of R and conversely every homomorphic image S of R under φ is a factor ring $R/\ker\varphi$. Moreover R/I is an integral domain if and only if I is a prime ideal. So your question is: If R is an integral domain that satisfies ACC on principal ideals and if P is a prime ideal of R must R/P have ACC on Principal ideals? The answer is: Not necessarily. Of course if R is a Noetherian domain and P is a prime ideal of R then R/P is Noetherian and hence has ACCP, but generally the situation is drastically different.

You will need to know about rings of polynomials in several variables. Zariski and Samuel [ZS, Commutative Algebra, Volume 1, Von Nostrand 1958, pages 34-41] may provide the necessary background, though any graduate level text in Commutative ring Theory may suffice.

Let F be a field and let $\{X_i\}_{i\in N}$ be a set of indeterminates over F labeled by natural numbers, 1,2,3,.... We can regard $F[\{X_i\}_{i\in N}] = \cup_{n\geq 1}F[X_1, X_2, ..., X_n]$, because a polynomial is a finite sum of monomials and each monomial is of the form $rX_1^{a_1}X_2^{a_2}...X_n^{a_n}$, a finite product, where $r \in F$ and $a_i \geq 0$. Also recall from [ZS, page 37], Theorem 12, which can be stated, assuming that our rings have 1 ($\neq 0$), as: Let S be a polynomial ring in the elements $x_1, x_2, ..., x_n$, over a ring R, let Δ be a unitary algebra over R and let $y_1, y_2, ..., y_n$ be elements of Δ.Then there is unique homomorphism φ from S onto $R[y_1, y_2, ..., y_n]$ such that $\varphi(x_i) = y_i$. Moreover φ is an isomorphism if and only if y_i are algebraically independent over R.

The above Theorem can be extended to the case of $F[\{X_i\}_{i\in N}]$ because $F[\{X_i\}_{i\in N}] = \cup_{n\geq 1}F[X_1, X_2, ..., X_n]$.

This can be done without much trouble using the ideas involved in the proof of the above theorem. Let $\{y_i\}_{i\geq 1}$ be a set of elements of a suitable ring Δ containing F. Define $\varphi : F[\{X_i\}_{i\in N}] \rightarrow F[\{y_i\}_{i\geq 1}]$ by $\varphi(f(X_{i1}, ..., X_{ir}) = f(y_{i1}, ..., y_{ir})$. Note that it is like evaluating the function f at the point $(y_{i1}, ..., y_{ir})$. Now let $f, g \in F[\{X_i\}_{i\in N}]$. Then there exists a natural number n such that $f, g \in F[X_1, X_2, ..., X_n]$. That φ is a homomorphism can be easily seen because $\varphi(f+g) = (f+g)(y_1, y_2, ..., y_n) = f(y_1, y_2, ..., y_n)+g(y_1, y_2, ..., y_n) = \varphi(f)+\varphi(g)$ and of course $\varphi(fg) = fg(y_1, y_2, ..., y_n) = f(y_1, y_2, ..., y_n)g(y_1, y_2, ..., y_n) = \varphi(f)\varphi(g)$. That this function is unique, such that $\varphi(X_i) = y_i$, and onto is obvious.

Next $F[\{X_i\}_{i\in N}]$ is a UFD and so satisfies ACCP. Let X be an indeterminate over F and let $S = \{X^q : q \in Q^+\}$. It is well known that the semigroup ring $F[S] = \{\sum_{i=1}^{n} f_iX^{q_i} : f_i \in F, q_i \in Q^+\}$ is a Bezout domain (see e.g. HD0308 at http://www.lohar.com/mithelpdesk/) that does not satisfy ACCP, because we have for any $q \in Q^+\backslash\{0\}$ the chain $(X^q) \subseteq (X^{\frac{q}{2}}) \subseteq (X^{\frac{q}{2^2}}) \subseteq ... \subseteq (X^{\frac{q}{2^n}}) \subseteq ...$

Take for Δ a ring that contains $F[S]$ as a subring and select for y_i the elements $X^{\frac{1}{i}}$

Then $\varphi : F[\{X_i\}_{i\in N}] \rightarrow F[S]$ defined by $\varphi(f(X_{i_1}, X_{i_2}, ..., X_{i_n})) = f(y_{i_1}, y_{i_2}, ..., y_{i_n})$

where $y_j = X^{\frac{1}{j}}$, (by what we have seen above) is a homomorphism from $F[\{X_i\}_{i\in N}]$ onto $F[\{X^{\frac{1}{j}}\}_{j\geq 1}] = F[S]$. Now as $F[S]$ is an integral domain which is not a field $Ker(\varphi) = P$ must be a non-maximal prime ideal. Next $F[\{X_i\}_{i\in N}]/P \cong F[S]$. Now $F[\{X_i\}_{i\in N}]$ has ACCP as we have already mentioned and P is a prime ideal of $F[\{X_i\}_{i\in N}]$ such that $F[\{X_i\}_{i\in N}]/P$ does not satisfy ACCP.

Since the purpose here was to give a concrete example, that only involves straight-forward manipulations, we have used a very limited approach. In general if Γ is any infinite set and R any commutative ring then $R[\{X_\gamma : \gamma \in \Gamma\}] = \cup R[\{X_i : i \in I\}]$ taken over all finite subsets I of Γ. So, again each polynomial involves only finitely many variables and the sum and multiplication of polynomials can be carried as if they were polynomials in a polynomial ring in finitely many variables. This of course makes $R[\{X_\gamma : \gamma \in \Gamma\}]$ into a ring. Next let S be a ring containing R as a subring and let $H = \{s_\gamma \in S : \gamma \in \Gamma\}$. The $X_\gamma \mapsto s_\gamma$ gives the R-homomorphism $\varphi : R[\{X_\gamma : \gamma \in \Gamma\}] \to S$. This homomorphism is called the substitution map; the mechanics is similar to what we have already seen. Clearly $\varphi(R[\{X_\gamma : \gamma \in \Gamma\}]) = R[\{s_\gamma \in S : \gamma \in \Gamma\}]$ and so φ maps $R[\{X_\gamma : \gamma \in \Gamma\}]$ onto $R[\{s_\gamma \in S : \gamma \in \Gamma\}]$. Now how do we use this? For our type of examples we can restrict ourselves to domains with zero characteristic. So, pick a domain D with zero characteristic (that does not have ACCP). Note that such a domain has a copy of Z the ring of integers as a subring (we can identify that "copy" with Z) . Use D as an index set (in place of Γ), fix a set of indeterminates $\{X_d : d \in D\}$ over Z and for $Z[\{X_d : d \in D\}]$. Further set $H = \{d \in D\}$. Define the substitution map by $X_d \mapsto d$. By the above procedure $\varphi : Z[\{X_d : d \in D\}] \to Z[\{d \in D\}]$ is the substitution map assigning d to X_d for each $d \in D$. Now clearly $Z[\{d \in D\}] = D$ which is a domain that is not a field, because D does not satisfy ACCP. So $Ker(\varphi) = P$ is a non-maximal prime ideal and indeed $Z[\{X_d : d \in D\}]/P \cong D$. But $Z[\{X_d : d \in D\}]$ is a UFD and hence satisfies ACCP. So there are as many examples as you want of domains with ACCP with quotient rings that are domains that do not satisfy ACCP.

If the domain D without ACCP has characteristic $p > 0$, then D is an algebra over (a copy of) $F_p = Z/(p)$ and using the above procedure we can conclude that $D = F_p[\{d \in D\}]$ is a homomorphic image of the UFD $F_p[\{X_d : d \in D\}]$.

Combining both cases (zero and positive characteristic) we can also conclude that every integral domain is a homomorphic image of a UFD. This remarkable fact is reminiscent of the fact that every group is a homomorphic image of a free group. (I am grateful to Dan and David Anderson for some useful discussions, over the phone.)

Tiberiu Dumitrescu, who was exposed to the above write up came up with the following easy example that does not need the elaborate considerations given above.

The example goes as follows:

Define $f : A = Z + xZ[y][x] \to Z + xZ[1/2][x] = B$ sending x to x and y to $1/2$. Clearly f, being a substitution map, is a homomorphism of A onto B. So,

$A/Ker(f) \cong B$. Since B is a domain $Ker(f)$ is a prime ideal. Now A has ACCP since A is a subring of the UFD $Z[x, y]$, and so the factorization of elements of A is restrained by the degrees of x and y and by Z. On the other hand B does not have ACCP since $(x) \subsetneq (x/2) \subsetneq (x/4) \subsetneq ...$.