QUESTION (HD 2003) What is a "t-class group" of an integral domain D and how do you compute it?

ANSWER: For t-class groups you would need a dose of star operations, as the t-, that comes before class group, is a star operation. (The following material, on star operations etc., is a modified version of some material in hd0901.pdf)

Let $F(D)$ denote the set of nonzero fractional ideals of D and let K be the quotient field of D. A star operation $*$ on D is a function $*\colon F(D) \to F(D)$ such that for all $A, B \in F(D)$ and for all $0 \neq x \in K$

(a) $(x)^* = (x)$ and $(xA)^* = xA^*$,

(b) $A \subseteq A^*$ and $A^* \subseteq B^*$ whenever $A \subseteq B$,

(c) $(A^*)^* = A^*$.

For $A, B \in F(D)$ we define $*$-*multiplication* by $(AB)^* = (A^*B)^* = (A^*B^*)^*$. A fractional ideal $A \in F(D)$ is called a $*$-*ideal* if $A = A^*$ and a $*$-ideal of *finite type* if $A = B^*$ where B is a finitely generated fractional ideal. A star operation $*$ is said to be of *finite character* if $A^* = \bigcup\{B^* \mid 0 \neq B$ is a finitely generated subideal of $A\}$. For $A \in F(D)$ define $A^{-1} = \{x \in K \mid xA \subseteq D\}$ and call $A \in F(D)$ $*$-*invertible* if $(AA^{-1})^* = D$. Clearly every invertible ideal is $*$-invertible for every star operation $*$, because $D^* = D$. If $*$ is of finite character and A is $*$-invertible, then A^* is of finite type. The most well known examples of star operations are: the v-operation defined by $A \mapsto A_v = (A^{-1})^{-1}$, the t-operation defined by $A \mapsto A_t = \bigcup\{B_v \mid 0 \neq B$ is a finitely generated subideal of $A\}$, and the d operation defined by $A \mapsto A$, for all $A \in F(D)$. Given two star operations $*_1, *_2$ we say that $*_1 \leq *_2$ if $A^{*_1} \subseteq A^{*_2}$ for all $A \in F(D)$. Note that $*_1 \leq *_2$ if and only if $(A^{*_1})^{*_2} = (A^{*_2})^{*_1} = A^{*_2}$. By definition t is of finite character, $t \leq v$ while $\rho \leq t$ for every star operation ρ of finite character; thus a v-ideal is a $*$-ideal for every star operation $*$ and so is a nonzero principal fractional ideal. The v-operation is also important in that, for any star operation $*$, and for A a $*$-invertible fractional ideal we have $A^* = A_v$ (see page 433 of Zafrullah's [46]).

If $*$ is a star operation of finite character then using Zorn's Lemma we can show that a proper integral ideal maximal w.r.t. being a star ideal is a prime ideal and that every proper integral $*$-ideal is contained in a maximal $*$-ideal. Let us denote the set of all maximal $*$-ideals by $*$-$max(D)$. It can also be easily established that for a star operation $*$ of finite character on D we have $D = \bigcap_{M \in *\text{-}\max(D)} D_M$. A v-ideal A of finite type is t-invertible if and only if A is t-locally principal i.e. for every $M \in t\text{-}\max(D)$ we have AD_M principal. An integral domain D is called a Prüfer v-multiplication domain (PVMD) if every nonzero finitely generated ideal of D is t-invertible. I recommend looking up sections 32 and 34 of Gilmer's book [32], and [46] for star operations.

For a given star operation $*$ define $Inv_*(D) = \{A \in F(D) | A$ is a $*$-invertible $*$-ideal$\}$. Since the $*$-multiplication of two $*$-invertible $*$-ideals is again a $*$-invertible $*$-ideal, the $*$-multiplication, in $Inv_*(D)$, is obviously associative, and since D is the identity and the inverse is assured by the definition of $Inv_*(D)$, we conclude that $Inv_*(D)$ is an abelian group under $*$-multiplication. Next let $P(D) = \{xD : x \in K\backslash\{0\}\}$ be the group of principal fractional ideals of D. Since each principal fractional ideal is a $*$-ideal for any star operation $*$, $P(D)$ is a subgroup of $Inv_*(D)$ for every star operation $*$. The quotient group $Inv_*(D)/P(D)$ is called the $*$-class group, $Cl_*(D)$. Now you can have a $*$-class group for any star operation that you fancy. If you set $* = d$ the identity operation that takes each $A \in F(D)$ to A, you have the d-class group which is clearly the ideal class group or the Picard group of D ($Pic(D)$). If, on the other hand, you put $* = v$ or $* = t$ you get the v-class group or the t-class group, respectively. (What is interesting, all these $*$-class groups are defined for a general integral domain.) As it usually happens, the notion of a t-class group appeared in a paper by A. Bouvier [17], much before the more general $*$-class group that was introduced by David Anderson in [10].

There is another class group that was known way before the t-class group which has some very important applications, in number theory and geometry. This group, for a (completely integrally closed) domain D, is called the divisor class group and is denoted by $Cl(D)$. To define it we need just the v-operation. (Gilmer has a more general definition in section 34 of [32] but we shall leave it as further reading.)

Define on $F(D)$ the relation ˜ as A˜B, if and only if $A_v = B_v$ (or equivalently $A^{-1} = B^{-1}$) for $A, B \in F(D)$. Obviously ˜ is an equivalence relation, usually called Artin's equivalence. In fact Van der Waerden calls ˜ "quasi-equality" at page 185 of [41]. If you read [41] remember that Van der Waerden was dealing only with integrally closed Noetherian domains. In any case after taking a few hints of historical nature let's get back to the task at hand.

For each $A \in F(D)$ denote by $div(A)$ the, divisorial, equivalence class of A, i.e. the set of all $X \in F(D)$ with $X_v = A_v$. Denote by $\mathcal{D}(D) = \{div(A) : A \in F(D)\}$. Define on $\mathcal{D}(D)$ the operation $+$ by $div(A) + div(B) = div(AB)$. It is easy to see that $\mathcal{D}(D)$ is a commutative semigroup with identity $div(D)$. Now by this definition, $A \in F(D)$ has an inverse if there is a $B \in F(D)$ such that $div(A)+div(B) = div(D)$. This of course requires that $div(AB) = div(D)$ which means AB˜D which happens if and only if $(AB)_v = D$ i.e. A is v-invertible. It can be shown that in this case $B_v = A^{-1}$ see e.g. page 433 of Zafrullah [46]. So, $\mathcal{D}(D)$ being a group requires that every $A \in F(D)$ is v-invertible. But it is well known that every $A \in F(D)$ is v-invertible if and only if D is completely integrally closed, as indicated in Theorem 34.3 of [32]. Thus $\mathcal{D}(D)$ is a group under $+$, as defined above, if and only if D is completely integrally closed. Now for D completely integrally closed define $\mathcal{P}(D) = \{div(A), A \in F(D) : div(A)$ contains a nonzero principal fractional ideal$\}$. It is easy to see that $\mathcal{P}(D)$ is a subgroup of $\mathcal{D}(D)$ under $+$. Now define the divisor class group as the quotient group $Cl(D) = \mathcal{D}(D)/\mathcal{P}(D)$. It is easy to see that if, for D completely integrally closed, we define $\varphi : \mathcal{D}(D) \to Inv_v(D)$ by $\varphi(div(A)) = A_v$ then $\varphi(div(A) + div(B)) = \varphi(div(AB)) = (AB)_v$ and φ is a group isomorphism, it being easy to see that φ is well-defined and is onto and one-one. Now $\mathcal{D}(D) \cong Inv_v(D)$ and in a similar fashion $\varphi(\mathcal{P}(D)) = P(D)$ the set of nonzero principal fractional ideals of D under multiplication. Thus, for a completely integrally closed integral domain D we can represent the divisor class group by $Inv_v(D)/P(D)$, the v-

class group $Cl_v(D)$. It may however be noted that, as we have seen above, the v-class group can be defined for any integral domain D and that $Cl_v(D)$ is a divisor class group only when D is completely integrally closed.

The divisor class group becomes very useful when D is a Krull domain; which is known to be completely integrally closed. Recall that D is a Krull domain if D is a locally finite intersection of localizations at height one prime ideals and if localization at each height one prime is a discrete valuation domain. A good source for divisor class groups is Robert Fossum's [29]. One among the many uses of the divisor class group of a Krull domain D is the fact that a Krull domain is a UFD if and only if $Cl(D) = Cl_v(D)$ is trivial. Thus a Krull domain D is a UFD if and only if every divisorial ideal of D is principal. If D is Krull and $Cl(D)$ is torsion we get a generalization of UFDs called Almost Factorial Domain (fast factoriell ringe), page 33 of [29]. In [29] it was shown that a Krull domain D is almost factorial if and only if for each pair $f, g \in D\backslash\{0\}$ there is a natural number n such that $f^nD \cap g^nD$ is principal. The almost factorial domains were introduced by Storch [40].

Now it is well known that D is a Krull domain if and only if every nonzero ideal of D is t-invertible (see e.g. Theorem 2.3 of Houston and Zafrullah's [33]. Also note that every t-invertible t-ideal of any domain is actually a v-invertible v-ideal and hence divisorial. Further because every divisorial ideal is actually a t-ideal and hence in a Krull domain every divisorial ideal is actually a t-invertible t-ideal. Putting these observations together we conclude that for D Krull, $Inv_t(D) = Inv_v(D)$ and so $Cl_t(D) = Cl(D)$.

A PVMD is considered a good generalization of both of Prufer domains (every nonzero finitely generated ideal is invertible) and Krull domains and the first impression was that the t-class group was a sort of generalization of the divisor class group. So the early impulse was to study it as such and find results corresponding to results on divisor class groups. Sure enough, for PVMDs the t-class group went smoothly and it was obvious that a PVMD D is a GCD domain if and only if $Cl_t(D)$ is trivial [17, Proposition 2]. But there was evidence that the t-class group could work for more general domains. Zafrullah [43] studied almost GCD (AGCD) domains as domains D such that for every pair $x, y \in D\backslash\{0\}$ there is a natural number $n = n(x, y)$ such that $x^nD \cap y^nD$ is principal. In [43] it was shown that a PVMD with torsion t-class group is AGCD and conversely an integrally closed AGCD domain is a PVMD with torsion t-class group. There was also a script [19] written by Bouvier based on my answers to his questions while he was working on [17] An improved version of that script appeared as Bouvier and Zafrullah's [20]. In [20] we studied the conditions under which $Cl_t(D)$ was torsion or trivial etc.. Driss Nour-El-Abidine and A. Ryckaert wrote theses under the supervision of Bouvier at Univ. Claude Bernard, Lyon, France studying t-class groups. Later Anderson and Zafrullah, in [8], decided that every AGCD domain has torsion t-class group.

On the other hand a domain D that satisfies ACC on integral divisorial ideals, called a Mori domain, is a generalization of both of Noetherian and Krull domains. Barucci and Gabelli studied the t-class groups of Mori domains. Their work is mentioned in David Anderson's second survey, [11]. The local class group is the group $Inv_t(D)/Inv(D)$ and it was Bouvier's work on the local class groups of Krull domains i.e. $\mathcal{D}(D)/Inv(D)$ [18] that prompted me to suggest to him to work on the t-class groups. (Soon after writing [20], Professor Alain Bouvier got involved in research and management aspects of education and teacher-training in France . He has since served his country in various high level capacities, including the rectorship of a university. To have a glimpse of his distinguished career the reader may look up:

http://faculty.kfupm.edu.sa/math/kabbaj/FezConf/BouvierCV.pdf

There has been a great deal of interest in the t-class groups and local class groups. Some of the contributors such as Marco Fontana, S. Gabelli and M.H. Park are mentioned in a recent paper by Anderson, Fontana and Zafrullah [15]. There you can also see the definition of the $*$-class group for the semistar operations and a decent introduction to semistar operations. In the Anderson-Fontana-Zafrullah paper you would also see, for the first time, a discussion of the v-class group of some integral domains. The notion of the t-class groups was also translated into the language of monoids by [34]. There are many more contributors to the study of the t-class groups, such as Said El Baghdadi, M. Khalis, S. Kabbaj, G. Chang. If you read the literature you will find many more names. Of course a lot of monoid-theorists, including Geroldinger have studied the t-class groups of monoids, with the same definition because what they call a "monoid" is a replica of the multiplicative monoid of an integral domain. But I stop here because I think I have given enough introduction to the notion of $*$-class groups and to the literature.

Now let's look at how one "computes" the $*$-class group, or specifically the t-class group.

The best answer to your question is the following. As in the case of the divisor class group of a completely integrally closed integral domain, or as in the case of an ideal class group of a domain, there is no set formula for computing the t-class group, or local class group, of an integral domain. You compute or decide on a t-class group, or local class group, of an integral domain D, by studying the properties of the t-invertible t-ideals of the integral domain, or construction, under consideration. The best I can do is to give you examples of how I have and how others have "computed" the class groups in question.

To take care of the basics, let's look at what kind of domains will have class group that is trivial or torsion. This was systematically, though briefly, done in [20]. Though of course the treatment was limited to the t-operation. Let's first check what we need to know in the general case.

For a general star operation $*$ the $*$-class group of an integral domain D is $Cl_*(D) = Inv_*(D)/P(D)$ where $Inv_*D)$ is the group of $*$-invertible $*$-ideals. Now as indicated in [46], every $*$-invertible $*$-ideal is a v-invertible v-ideal. So, for any star operation $*$ we have $Cl_*(D) = Inv_*(D)/P(D) \subseteq Inv_v(D)/P(D)$.

We can go a bit more general and note that if $*_1 \leq *_2$ are two star operations then $Cl_{*_1}(D) \subseteq Cl_{*_2}(D)$, [10]. But the trouble is that there are no known (or interesting) $*$-class groups between the ideal class group and the t-class group and similarly there seem to be no known $*$-class groups between $Cl_t(D)$ and $Cl_v(D)$, thanks to the rather peculiar properties of $*$-invertibility, see Theorem

1.1 of [46], that, say, among other things that, every $*$-invertible $*$-ideal is a v-invertible v-ideal and if $*$ is of finite character then every $*$-invertible $*$-ideal is a t-invertible t-ideal. Thus we have: For every star operation $*$ we have $Cl_*(D) \subseteq Cl_v(D)$ and if $*$ is of finite character, then $Cl_*(D) \subseteq Cl_t(D) \subseteq Cl_v(D)$. However, care must be taken, even though every $*$-invertible $*$-ideal is a v-invertible v-ideal, not every $*$-class group is a v-class group. For there are numerous examples of domains D such that $Cl_*(D) \subsetneq Cl_v$, and $Cl_*(D) \subsetneq Cl_t$ for some star operation $*$, see e.g. Section 2 of [15]. The section in [15] mentioned above provides a good study of how to compute the $*$-class group in certain situations. That section ends, by the way, with the following remark: Given a rank one valuation domain D, if we assume in Theorem 2.7 that D is not a DVR and that $G \neq \mathbb{R}$, then $Cl_v(D)$ (which coincides in this case with the divisor class group of D) is not zero zero, whereas the t-class group $Cl_t(D)$ $(= Pic(D))$ is zero. Having shown that both the divisor class and t-class group can coexist without being equal, we conclude that the t-class group is not a generalization of the divisor class group.

Now what does "$Cl_*(D)$ is trivial" mean? Obviously, because $Cl_*(D) = Inv_*(D)/P(D)$, "$Cl_*(D)$ is trivial" means that everything in $Inv_*(D)$ is in $P(D)$. That is "$Cl_*(D)$ is trivial" means every $*$-invertible $*$-ideal of D is a principal fractional ideal. On the other hand "$Cl_*(D)$ is torsion" means that for each ideal I in $Inv_*(D)$ there is a positive integer $n = n(I)$ such that $(I^n)^*$ is a principal fractional ideal. We may use $Cl_*(D) = (1)$ to denote "$Cl_*(D)$ is trivial" or if we are used to using the additive notation of the divisor class group we may use $Cl_*(D) = (0)$ to denote "$Cl_*(D)$ is trivial". There is no notation to indicate that "$Cl_*(D)$ is torsion". To indicate that $Cl_*(D)$ is finite we may use the standard notation: $|Cl_*(D)| < \infty$.

Now before we delve into examples here's another group which may come in handy, as we proceed. Associated to each integral domain D there is the group $G(D) = \{kD | k \in K\backslash\{0\}\}$ of nonzero principal fractional ideals. It is just the set $P(D)$ except that $G(D)$ is partially ordered by $kD \leq hD \Leftrightarrow kD \supseteq hD$. Now this partial order has another property, it is compatible with the group operation, multiplication of principal fractional ideals. That is if $x, y, z \in K\backslash\{0\}$ and $yD \leq zD$ then $xyD = xDyD \leq xDzD = xzD$. (Obviously this follows from the fact that $yD \supseteq zD \Rightarrow xyD \supseteq xzD$.) Next $mD = \sup(hD, kD)$ exists if and only if $mD \geq hD, kD$ and if there is an $lD \geq hD, kD$ then $lD \geq mD$. Now $mD \geq hD, kD \Leftrightarrow mD \subseteq hD \cap kD$ and "if there is an $lD \geq hD, kD$ then $lD \geq mD$" translates to "if there is an $lD \subseteq hD \cap kD$ then $lD \subseteq mD$. But this forces $mD = hD \cap kD$. In other words: For all $hD, kD \in G(D)$, $\sup(hD, kD)$ exists if and only if $hD \cap kD$ is principal. Now we know that in a partially ordered group, $\inf(x, y) = (xy)(\sup(x, y))^{-1}$. So $\inf(hD, kD) = (hDkD)(hD \cap kD)^{-1} = (\frac{1}{h}D \cap \frac{1}{k}D)^{-1} = ((hD + kD)^{-1})^{-1} = (hD + kD)_v$. Now we know that D is a GCD domain if and only if the intersection of every pair of nonzero principal (fractional) ideals is principal. Also a partially ordered group G is a lattice ordered group if and only if sup (and hence inf) of every pair of elements exists. Thus D is a GCD domain if and only if $G(D)$ is a lattice ordered group.

Let's take a few examples of how, in some cases, we can decide directly, on what the v-class, or t-class group is. Let's call a domain D v-local if D is local with the maximal ideal M a v-ideal. Take a v-invertible ideal I in D. Claim that I is invertible. For if not, then $D \neq II^{-1} \subseteq M$. But as I is v-invertible we have $D = (II^{-1})_v \subseteq M_v = M$, a contradiction. Thus, every v-invertible ideal of a v-local domain D is principal.

Working along the same lines, or following Proposition 2.10 of [28], we conclude that if D is a t-local domain, then every t-invertible t-ideal of D is principal. I record these observations in the following proposition.

Proposition 1 *Let (D, M) be a local domain. (a) If M is a v-ideal, $Cl_v(D) = (0)$ and (b) if M is a t-ideal, $Cl_t(D) = (0)$.*

Let's note that if in a local domain (D, M) M is a v-ideal, then M is a t-ideal and so, in this case, $Cl_t(D) = (0)$. Generally though, $Cl_v(D) = (0)$ implies $Cl_t(D) = (0)$ because $Cl_t(D) \subseteq Cl_v(D)$. Yet, on the other hand, if $Cl_t(D) = (0)$ we may not have $Cl_v(D) = (0)$, as mentioned above. To see this we have to take a sort of "North-Western route". In [44] I studied integral domains with the property that for all $A, B \in F(D)$ we have $(AB)^{-1} = A^{-1}B^{-1}$ and called them "generalized Dedekind domains" or "g-Dedekind domains". As it turned out, D is a g-Dedekind domain if and only if for all $A \in F(D)$ we have A_v invertible. It is easy to see that a g-Dedekind domain is completely integrally closed Corollary 1.4 of [44]. Now as every invertible ideal of a local domain is principal we conclude that if a local domain (D, M) is a g-Dedekind domain, then $Cl_v(D) = (0)$.Let's throw in the t-local property and conclude at the following result.

Proposition 2 *A t-local g-Dedekind domain (D, M) is a rank one valuation domain.*

Proof. Indeed, as a g-Dedekind domain is a PVMD [44] and a t-local PVMD is a valuation domain [28], we conclude that a t-local g-Dedekind domain is a valuation domain. Now to decide that the rank of D is one we can use the fact that a g-Dedekind domain is completely integrally closed.

Next let us read the last paragraph on page 292 of [44]. (I am copying it almost verbatim here.) Let G be a totally ordered group and let $G^+ = \{g \in G | g \geq 0\}$ denote the positive cone of G. A subset H of G^+ is called an upper class of positive elements if for all $h \in H$, $k > h$ implies $k \in H$. Let K be a field, v be a valuation on K and let V be the valuation ring associated to v. That is let $V = \{x \in K | v(x) \geq 0\}$. Next let $G(V)$ be the group of divisibility or the value group $= \{kV | k \in K \backslash \{0\}\}$ of V, ordered by $kV \geq hV \Leftrightarrow kV \subseteq hV$. (The upper classes are just like Dedekind cuts.) It is well known that there is a one to one correspondence between the set of nonzero integral ideals $I(V)$ of V and $C(G(V))$ the set of upper classes of positive elements of $G(V)$ (see e.g. page 10 of [39]. This correspondence is given by $A \mapsto V(A) = \{Va | a \in A \backslash \{0\}\}$. (Of course A being principal corresponds to $V(A)$ containing its least element.) If v is of rank one $G(V)$ can be regarded as a subgroup of the group of real numbers, Theorem 1, p. 6 of [39]. A lattice ordered group G is said to be complete if every subset S of G that is bounded from below has a greatest lower bound, inf (and dually) belongs to G. Now $G(V)$ being totally ordered, is lattice ordered and so its being complete or non-complete can be considered. Obviously if v is of rank one, for every nonzero integral ideal A of V, $V(A)$ is bounded from below and hence has an infimum in $\mathbb{R}$, the set of real numbers, and this infimum may or may not belong to $G(V)$. If, however, $G(V)$ is complete, $\inf(V(A))$ belongs to $G(V)$, for each nonzero ideal A of V. The above considerations lead to the following proposition.

Proposition 3 *A t-local g-Dedekind domain is a rank one valuation domain V with complete value group $G(V)$. Conversely if V is a rank one valuation domain with complete value group, then V is a g-Dedekind domain.*

For the first part Proposition 2 and the above discussion can be used and for the converse, Theorem 2.6 of [44].

Corollary 4 *For a rank one valuation domain V, $Cl_v(V) = 0$ if and only if V is a g-Dedekind domain.*

Corollary 4 has some very interesting applications. Note that if V is a rank one valuation such that $G(V)$ is not complete then by Corollary 4, $Cl_v(V) \neq (0)$, yet as V is t-local $Cl_t(V) = (0)$. (Compare with the remark at the end of Section 2 of [15].)

Example 5 *Let V be a rank one valuation such that $G(V) = Q$ the set of rationals. Then $Cl_v(V) \neq (0)$ yet $Cl_t(v) = (0)$.*

Now, noting that, when V is a discrete valuation of rank one, $G(V) = Z$, and so is complete. Also when V is of rank one such that V is non-discrete with $G(V)$ not complete, as in Example 5, we have $Cl_v(V) \neq (0)$. Now we know that when V is of rank one V is completely integrally closed and so $Cl_v(D)$ is actually the divisor class group $Cl(V)$ of V. Thus if, say, V is a rank one valuation domain such that $G(V) = Q$, we have both $Cl(V)$ and $Cl_t(V)$ defined for V and distinct in that $Cl_t(V) = (0)$ and $Cl(V) \neq (0)$. Consequently we have the following result.

Theorem 6 *The t-class group of an integral is a concept distinct from that of the divisor class group.*

Corollary 7 *The following assertion in [10]: "Probably the best generalization of divisor class group to an arbitrary integral domain is given by the t-class group." is false in that the t-class group is not a generalization of the divisor class group.*

Remark 8 *The g-Dedekind domains, by the way, were also studied, under the name of pseudo Dedekind domains, in a later paper [4] by Dan Anderson and B.G. Kang.*

While it is the case that for a v-local domain D, we have $Cl_v(D) = (0)$; it is generally not the case that if D is locally a v-local domain then $Cl_v(D) = (0)$. Same holds if we replace $Cl_v(D)$ with $Cl_t(D)$. The example in both cases is a Dedekind domain.

It was indicated in Proposition 3.14 of [15] that if D is a v-domain and $Cl_v(D) = (0)$, then D must be a GCD domain. However, in general, if $Cl_v(D) = (0)$, D may not even be pre-Schreier. Indeed D does not even have to be a, so-called, $*$-domain.

Recall that D is a pre-Schreier domain if, for all $x, y, z \in D\backslash\{0\}$, $x|yz$ implies $x = rs$ where $r|y$ and $s|z$. It was shown in [45] that D is a pre-Schreier domain if and only if D satisfies

$* : ((\cap(a_i))(\cap(b_j)) = \cap_{ij}(a_ib_j)$ for all $a_1, ..., a_n, b_1, ...b_m \in D$

along with the property that $x \in ((\cap(a_i))(\cap(b_j)) = \cap_{ij}(a_ib_j)$ implies that $x = rs$ where $r \in ((\cap(a_i))$ and $s \in (\cap(b_j))$. Obviously the property $*$ is much weaker than the pre-Schreier property. A still weaker property

$** : ((\cap(a_i))(\cap(b_j)) = \cap_{ij}(a_ib_j)$ for all $a_1, a_2, b_1, b_2 \in D$

was shown in [AAJ] to make a Noetherian domain locally factorial. Next, a one dimensional Noetherian local domain that is not a valuation domain is a v-local domain that does not have even the $**$ property. Thus we have the following statement.

Proposition 9 *(1) Suppose that D is a v-domain. If $Cl_v(D) = (0)$, then D is a GCD domain. The converse does not hold. (2) Generally if $Cl_v(D) = (0)$, then D does not have to satisfy the $**$ property, one of the most general forms of the GCD property.*

One way of "computing" the v-class group group or the t-class group is to show that it is the ideal class group etc.

Proposition 10 *(i) If D is a g-Dedekind domain, then $Cl_v(D) = Cl_d(D)$, (ii) if D is a $*$-domain, then $Cl_t(D) = Cl_d(D)$ and (iii) If D is a g-Dedekind domain, then $Cl_v(D) = Cl_t(D) = Cl_d(D)$.*

Proof. Recall that D is a g-Dedekind domain if and only if for all nonzero $A \in F(D)$ we have A_v invertible. So, $Inv_v(D) = Inv(D)$ and this gives $Cl_v(D) = Inv_v(D)/P(D) = Inv(D)/P(D) = Cl_d(D)$, (ii) we show that a t-invertible t-ideal is an invertible ideal in a $*$-domain. The main tool behind this conclusion is that a t-invertible t-ideal I is expressible as a finite intersection of principal ideals. For I is t-invertible if and only if I_v is of finite type, $(II^{-1})_v = D$ and I^{-1} is a v-ideal of finite type [46], Theorem 1.1. Now if $I = (a_1, ...a_n)$ is a t-invertible ideal in a domain with property $*$, then $I_v = \cap_1^m x_iD$, $I^{-1} = \cap_1^n y_jD$ and the $*$-property give us $I_vI^{-1} = \cap_{ij}x_iy_jD$. Since the right hand side in the preceding equation is a v-ideal, we get $I_vI^{-1} = D$. For (iii) note that D being g-Dedekind $\Leftrightarrow \forall_{A,B\in F(D)}((AB)^{-1} = A^{-1}B^{-1}) \Rightarrow \forall_{A,B\in f(D)}((AB)^{-1} = A^{-1}B^{-1}) \Leftrightarrow D$ satisfies $*$.

The real computation of the t-class group that took place was in section 3 of a paper by Stefania Gabelli [30]. In it she proved as Proposition 3.2 the following result.

Proposition 11 *Let D be an integral domain and let $\Delta(D)$ denote the set of divisorial ideals of D. Then: (1) the correspondence $\varphi : \Delta(D) \to \Delta(D[X])$ defined by $\varphi(J) = JD[X]$ is an injective homomorphism of semigroups. Moreover J is a t-invertible t-ideal of D if and only if $JD[X]$ is a t-invertible t-ideal of $D[X]$, (2) φ induces an injective homomorphism of the groups $\psi : Cl_t(D) \to Cl_t(D[X]._i$*

Then she nails a remarkable result in Theorem 3.6, that I state as the following result.

Theorem 12 *Let $\psi : Cl_t(D) \to Cl_t(D[X]$ be the homomorphism defined in Proposition 11 then ψ is an isomorphism if and only if D is integrally closed.*

The proofs of Lemma 3.5 and Theorem 3.6 of [30] are a good study, if you want to work with class groups.

Let's get back to the business at hand. Let's recall that if D is an integral domain with quotient field K. The group of divisibility $G(D)$ of D is a partially ordered group that is a lattice ordered group if and only if D is a GCD domain. As we have seen $P(D)$ is just like $G(D)$ and looking at [46],..., $*$-$Inv(D)$ too is like $G(D)$, in some ways. One may want to look into locally finite intersections of localizations at prime ideals. Let us call D an LFIP domain if D has a family $F = \{P_{\alpha \in I}\}$ of prime ideals such that $D = \cap D_{P_\alpha}$ and each nonzero non unit of D is a unit in D_{P_α} for almost all $\alpha \in I$. Let's call F the LFIP-defining family of D.

Let's call an element of D primary if xD is a primary ideal. Recall that a domain D is called a Weakly Factorial Domain (WFD) if every nonzero non unit of D is expressible as a finite product of primary elements. This notion was introduced by Anderson and Mahaney in [5]. The class group of a weakly factorial domain is zero, because in a weakly factorial domain, every t-invertible t-ideal is principal. The proof may be a little hard to reconstruct from [7], so we include it in the result below. For this recall from [7] that a weakly factorial domain D is weakly Krull, i.e., D is a locally finite intersection of localizations at height one prime, each of which is a maximal t-ideal and that D is weakly factorial if and only if the following holds: If P is a prime ideal of D minimal over a proper principal ideal (x), then ht$P = 1$ and $xD_P \cap D$ is principal.

Proposition 13 *Let D be a weakly factorial domain. Then every t-invertible t-ideal I of D is principal.*

Proof. Let I be a t-invertible t-ideal in a WFD D. Then $I = (x_1, ..., x_r)_v$. Because each of the x_i belongs to at most a finite number of height one primes of D, I belongs to at most a finite number say $P_1, P_2, ..., P_s$ of height one primes of D. Next, because I is a t-invertible t-ideal and each P_i is a maximal t-ideal we have $ID_{P_i} = y_i D_{P_i}$ and of course as $y_i D_{P_i} \cap D = (z_i)$ we have $ID_{P_i} \cap D = (z_i)$. Since I is divisorial we have $I = \cap_{Q \in X^1(D)} ID_Q = ID_{P_1} \cap ... \cap ID_{p_S} \cap (\cap_{Q \notin \{P_1,...,P_s\}} D_Q) = ID_{P_1} \cap D ... \cap ID_{p_S} \cap D = (z_1) \cap (z_2) ... \cap (z_s)$. Since (z_i) are mutually t-comaximal, we have $I = I_t = (\prod_{i=1}^{s} (z_i))_t = (\prod_{i=1}^{s} z_i)$, principal.

Using Theorem 3.1 of [6] one can show that D is a weakly Krull domain if and only if (A) Every proper t-invertible t-ideal of D is a t-product of primary t-ideals. (Setting (A) as (4') in Theorem 3.1 of [6], the proof will go as follows: (4) $\Rightarrow$ (4') because (4') is a special case of (4) and (4') $\Rightarrow$ (3) as a special case. But (3) $\Leftrightarrow$ (4) being equivalent, according to Theorem 3.1 of [6].) Also according to Theorem 3.1 of [6], one has that D is a weakly Krull domain if and only if "Whenever P is a prime ideal minimal over a proper principal ideal (x) we have $xD_P \cap D$ t-invertible." Thus we have the following result.

Proposition 14 *Let D be a weakly Krull domain. Then the following are equivalent.*

(1) D is weakly factorial,

(2) For every height one prime ideal P of D and for every $x \in P$ we have $xD_P \cap D$ principal,

(3) $Cl_t(D) = (0)$.

((1) $\Rightarrow$ (2) being a special case of (6) of Theorem of [7], (2) $\Rightarrow$ (3) can be reconstructed from the proof of Proposition 13. (3) $\Rightarrow$ (1) follows from the facts that in a weakly Krull domain for each nonzero non unit x we have $(x) = (\prod_{i=1}^{n}(xD_{P_i} \cap D))_t$, where P_i ranges over the height one prime ideals containing x, each of $xD_{P_i} \cap D$ is a t-invertible t-ideal and for each i, $xD_{P_i} \cap D$ is a primary ideal. Applying (3) gives $xD_{P_i} \cap D$ principal for each i.)

Call D an almost weakly factorial domain (AWFD) if for each nonzero non unit x of D, there is a positive integer $n(x)$ such that $x^{n(x)}$ is a product of primary elements. Theorem 3.4 of [6] shows that D is an AWFD if and only if D is a weakly Krull domain with torsion t-class group. Some of these ideas were further developed in [14].

The other area where "computing" or finding out the class group is to do with various polynomial ring constructions. Of these the $A + XB[X]$ is the foremost and the hardest to handle. To give you an ideal I have chosen a paper by David Anderson, Said El Baghdadi and Salah Kabbaj, [12].

It appears that grading is the best approach, in computing the class group of $A + XB[X]$ and that too in some special cases.

A commutative and cancellative monoid, usually written additively, with neutral element 0, is called a grading monoid. A grading monoid Γ is torsionless if $m\gamma = m\gamma'$, for some positive integer m, implies $\gamma = \gamma'$. An additive monoid M is partially ordered (or endowed with a partial order) if there is a reflexive transitive and antisymmetric relation $\leq$ defined on M such that $x \leq a$ implies $x + y \leq a + y$ for all $x, a, y \in M$. (In this case we say that $\leq$ is compatible with $+$.) A partially ordered monoid M is said to be endowed with a total order if for all $x, y \in M$ we have $x \leq y$ or $y \leq x$. According to Northcott [37], which is a good source on grading monoids, A grading monoid Γ can be endowed with a total order compatible with its structure as a monoid if and only if it is torsionless.

A ring R is called graded (or more precisely, Γ-graded) if there exists a family of subgroups $\{R_\gamma\}_{\gamma \in \Gamma}$ of R such that

(1) $R=\bigoplus_\gamma R_\gamma$ (as abelian groups), and (2) $R_\alpha\ R_\beta \subseteq R_{\alpha+\beta}$ for all $\alpha,\beta\in\Gamma$.

A non-zero element $x\in R_\gamma$ is called a homogeneous element of R of degree γ, $0\neq x\in R_\gamma$ is usually represented by x_γ. Note that if $R=\bigoplus_\gamma R_\gamma$ is a graded ring, then R_0 is a subring of R, $1\in R_0$ and R_γ is an R_0-module for all γ. We usually assume that $R_\gamma\neq(0)$ for each γ. The reason is that if we denote by $Supp(R)$ the set $\{\alpha\in\Gamma|R_\alpha\neq(0)\}$, then $Supp(R)$ is a submonoid of Γ, with the same properties.

By a (Γ-)graded integral domain $D=\bigoplus_{\gamma\in\Gamma}D_\gamma$ we mean an integral domain graded by an arbitrary torsionless grading monoid Γ. Thus, every $0\neq f\in R$ can be written as $f=x_{\alpha_1}+x_{\alpha_2}+...x_{\alpha_n}$ with $\alpha_1<\alpha_2<...<\alpha_n$. The x_{α_i} are the (homogeneous) components of f. Let H be the saturated multiplicative set of nonzero homogeneous elements of D. Then

$D_H=\bigoplus_{\gamma\in\Gamma}(D_H)_\gamma$, called the homogeneous quotient field of D.

An ideal I of a graded domain is homogeneous if (and only if) whenever $f\in I$ all homogeneous components of f belong to I. Thus an ideal I of a graded domain is homogeneous if and only if it is generated by homogeneous elements. Given a graded ring $R=\bigoplus_\gamma R_\gamma$ an R module M is graded if we can write $M=\bigoplus_\gamma M_\gamma$ where M_γ are subgroups such that $R_\gamma M_\delta\subseteq M_{\gamma+\delta}$ for all $\gamma,\delta\in\Gamma$. A submodule N of a graded module M is homogeneous, as in the case of ideals, if whenever $n\in N$ all homogeneous components of n belong to N. While the fractional ideals are defined in the usual manner, a homogeneous fractional ideal I is a homogenous submodule of $D_H=\bigoplus_{\gamma\in\Gamma}(D_H)_\gamma$ such that rI is a homogenous ideal of $D=\bigoplus_{\gamma\in\Gamma}D_\gamma$. If I and J are nonzero homogeneous fractional ideals, then $I:J$ is also a homogeneous fractional ideal and $I:_K J=I:_{D_H}J$. In particular, if I is homogeneous, then so is I_v.

The following results come from [12], some of them verbatim.

Lemma 15 *Let $R=A+XB[X]$ with B integrally closed. Let I be a homogeneous divisorial ideal of R, J the ideal of B generated by coefficients of all the polynomials of I. Let n be the least integer k such that $aX^n\in I$ for some nonzero $a\in B$ and let $W\subseteq J$ be the A-module consisting of all $a\in B$ such that $aX^n\in I$. Then J is a divisorial ideal of B and $I=X^nW+X^{n+1}J[X]$.*

[Thus if I is an integral divisorial ideal of $R=A+XB[X]$, B integrally closed, with $I\cap A=H\neq(0)$, then $I=H+XJ[X]$ where J is a divisorial ideal of B containing H.]

Lemma 16 *Let $R=A+XB[X]$ with B integrally closed. If I is a divisorial ideal of R, then there is a homogeneous ideal J of R such that $I=u(J)$ for some $u\in K[X,X^{-1}]$.*

The reason why B must be integrally closed is apparent from the following result.

Theorem 17 *Let $R=A+XB[X]$. Then B is integrally closed if and only if for every divisorial ideal I we have $I=u(W+XJ[X]$ where J is a v-ideal of B and $W\subseteq J$ a nonzero A-module.*

Lemma 18 *Let $R = A + XB[X]$. Let F_1 be a nonzero fractional ideal of A and F_2 a fractional ideal of B such that $F_1 \subseteq F_2$. Then $F_1 + XF_2[X]$ is a fractional ideal of R and we have $(F_1 + XF_2[X])^{-1} = F_1^{-1} \cap F_2^{-1} + XF_2^{-1}[X]$.*

Lemma 19 *Let $R = A + XB[X]$. Then $B[X]$ and $XB[X]$ are divisorial ideals of R.*

Theorem 20 *Let $R = A + XB[X]$ with B integrally closed. If I is a fractional v-invertible v-ideal of R, then $I = u(J_1 + XJ_2[X])$ for some $u \in qf(R)$, J_2 is a v-invertible v-ideal of B and J_1 is a nonzero ideal of A such that $J_1 \subseteq J_2$.*

Corollary 21 *Let $R = A + XB[X]$ with B integrally closed. If I is a fractional t-invertible t-ideal of R, then $I = u(J_1 + XJ_2[X])$ for some $u \in qf(R)$, J_2 is a t-invertible t-ideal of B and J_1 is a nonzero ideal of A such that $J_1 \subseteq J_2$.*

A fractional ideal I of $R = A + XB[X]$ is said to be extended from A if $I = u\,\mathrm{J}R$ for some $u \in qf(R)$ and some ideal J of A.

Lemma 22 *Let $R = A + XB[X]$ with B integrally closed and let I be a fractional divisorial ideal of R. Then $IB[X]$ is a divisorial ideal of $B[X]$ if and only if $I = uWR$ for some A submodule W of B and and some $u \in qf(R)$.*

Lemma 23 *Let $R = A + XB[X]$. Let I be a divisorial ideal of R of the form $I = J_1 + XJ_2[X]$ where J_1 is an ideal of A and J_2 an ideal of B with $J_1 \subseteq J_2$. Then the following statements are equivalent.*

(1) I is extended from A, (2) $J_2 = J_1B$ and (3) $IB[X]$ is divisorial in $B[X]$.

Theorem 24 *Let $R = A + XB[X]$. Let I be a fractional v-invertible v-ideal of R. Then TFAE: (1) I is extended from A and (2) $IB[X]$ is a divisorial ideal of $B[X]$.*

Lemma 25 *Let $R = A + XB[X]$. Then R is a flat A-module if and only if B is a flat A-module.*

Lemma 26 *Let $S \subseteq T$ be an extension of integral domains such that T is a flat S-module. If I is a nonzero finitely generated ideal of S, then (a) $(IT)^{-1} = I^{-1}T$ and (b) $(IT)_v = (I_vT)_v$.*

(Let me recall, results like this started appearing after the appearance of Lemma 4 of [42]. The said lemma was Lemma 26 for T a ring of fractions of S.)

Lemma 27 *Let $S \subseteq T$ be an extension of integral domains such that T is a flat S-module. If I is a v-ideal of finite type of S, then $(IT)^{-1} = I^{-1}T$.*

Proof. Let B be a finitely generated ideal such that $I = B_v$. Then $(IT)_v = (B_vT)_v = (BT)_v$. This gives $(IT)^{-1} = (BT)^{-1} = B^{-1}T = (B_v)^{-1}T = I^{-1}T$.

(Note that if the ring T is a flat S module and A is a t-ideal of T such that $A \cap S \neq (0)$, then $A \cap S$ is a t-ideal of S. For if $a_1, ..., a_n \in A \cap S$, then

$(a_1, ..., a_n)T \subseteq A$. Since A is a t-ideal, we have $((a_1, ..., a_n)T)_v \subseteq A$. But, by Lemma 26, $((a_1, ..., a_n)_v T)_v$ and so $(a_1, ..., a_n)_v T \subseteq ((a_1, ..., a_n)T)_v \subseteq A$. This forces $(a_1, ..., a_n)_v \subseteq A \cap S$. This is the most general form of Lemma 3.17 of [35] which, by the way, says that if I is a t-ideal of a ring of fractions R of D, then $I \cap D$ is a t-ideal.)

Recall that in an extension $A \subseteq B$ of domains, $B \in$ is said to be t-linked over A if for each finitely generated ideal I of A, $I^{-1} = A$ implies $(IB)^{-1} = B$.

Lemma 28 *Let $R = A + XB[X]$ and let J be an ideal of A. (1) If $(JR)_v = R$, then $J_v = A$ and (2) If $(JR)_t = R$, then $J_t = A$. Also (3) if B is a flat A-module and $J_t = A$, then $(JR)_t = R$. ((3) because B being a flat A-module makes $A + XB[X]$ flat and hence a t-linked extension of A, [3].*

Proposition 29 *Let $R = A + XB[X]$ such that B is a flat A-module and let J be an ideal of A. Then J is a t-invertible t-ideal of A if and only if JR is a t-invertible t-ideal of R.*

Theorem 30 *Let $R = A+XB[X]$ such that B is a flat A-module and integrally closed and let I be a fractional t-invertible t-ideal of R. Then $IB[X]$ is a (proper) divisorial ideal of $B[X]$ if and only if $I = u\,\mathrm{J}R$ for some $u \in qf(R)$ and some (proper) t-invertible t-ideal J of A.*

Lemma 31 *Let $R = A + XB[X]$ such that B is a flat A-module. Then $B[X]$ is a flat R module if and only if B is an overring of A.*

Lemma 32 *Let $R = A + XB[X]$ such that B is a flat A-module. Then the canonical map $\varphi : Cl_t(A) \to Cl_t(R)$ defined by $[J] \mapsto [JR]$ is a well-defined map and it is an injective homomorphism.*

(As B is a flat A-module R is a flat A-module and hence $A \subseteq A+XB[X]$ is a t-linked extension of [3]. Thus by Theorem 2.2 of [3], there is a homomorphism $\overline{\theta} : Cl_t(A) \to Cl_t(R)$ defined by $[IA] \mapsto [(IR)_t]$. Since I is a t-invertible t-ideal, I is a finite intersection of principal fractional ideals. Couple it with the fact that R is a flat A-module to get $(IR)_t = IR$. That the homomorphism is injective is now straightforward.)

The next two results seem to be what the authors were planning to get.

Theorem 33 *Let $R = A + XB[X]$ such that B is integrally closed and a flat overring of A. Then $Cl_t(A) \cong Cl_t(A + XB[X])$.*

Corollary 34 *Let S be a multiplicatively closed subset of A. If A is integrally closed, then $Cl_t(A + XA_S[X]) \cong Cl_t(A)$.*

A slightly improved corollary will be the following result.

Let S be a multiplicatively closed subset of A such that A_S is integrally closed, then $Cl_t(A + XA_S[X]) \cong Cl_t(A)$.

The authors of [12] go on to indicate various other, important, applications of Theorem 33, but I would stop here to make the following points.

(1) When "computing" a class group concentrate on integral t-invertible t-ideals, the definition ensures that the class of each t-invertible t-ideal should contain an integral ideal.

(2) I have followed [12], more or less result by result, until Theorem 33, to press home the fact that while every integral domain has the t-class group, not all t-class groups may be easy to find.

(3) In my opinion Lemma 32 presents the best scenario, in the $A + XB[X]$ situation. in that it shows that $Cl_t(A)$ is isomorphic to a subgroup of $Cl_t(A + XB[X])$.

(4) Looking at Corollary 34, I wonder if the $D + XD_S[X]$ construction of [25] had not come about, how would the consideration of the $A + XB[X]$ and how would a lot of activity on pullbacks come about?

(5) A consequence Theorem 33 is that if L is an extension field of $qf(A)$, then $Cl_t(A+XL[X]) \cong Cl_t(A)$. Most of the legwork for this result was done in [16] which was one of the earlier papers written on the class group. In [16] the authors consider the $D+M$ construction, that is if $T = K+M$ is a domain with K a field and D is a subring of K, they consider $R = D + M$. Two examples of the $D + M$ construction are $D + XL[X]$, and $D + XL[[X]]$, where L is an extension field of $qf(D)$. Proposition 2.4 of this paper can be a useful result to know and so is included below.

Proposition 35 *Let $T = L + M$ and $R = D + M$ be integral domains with D a subring of L. Let I be a nonzero fractional ideal of D. Then*

(1) $(I + M)^{-1} = I^{-1} + M$.

(2) $(I + M)_v = I_v + M$.

(3) I is finitely generated (resp., v-finite) if and only if $I + M$ is finitely generated

(resp., v-finite).

(4) $I \in Inv_t(D)$ if and only if $I + M \in Inv_t(R)$.

The computation of the t-class group or of the $*$-class group of a pullback is essentially similar to the computation of the class group of the $D + M$ and that of the $A + XB[X]$ domains. The papers that may be of interest in this area are [22], [26], [27] [23] etc. and If I have strength I might describe some of their results, though I think Chang's paper offer's the essence of the pullback approach.

When it comes to computing $Cl_t(D_S)$, given that $Cl_t(D)$ is known, the question of computing the class group becomes very interesting indeed. While, as we have seen, a t-class group is is not a generalization of a divisor class group the various similarities lead one to look into known results on divisor class groups to get a handle on the t-class group. Nagata proved that if D is a Krull domain, S is a multiplicative set in D and if $Cl(D)$ represents the divisor class group of D, then the map $f_S : Cl(D) \to Cl(D_S)$ defined by $[I] \mapsto [ID_S]$ is surjective and if S is generated by primes then f_S is an isomorphism [29], Corollaries 7.2 and 7.3. Just like the class groups, the PVMDs grew up as a sort of generalization of Krull domains, see [36]. As mentioned in [31], Proposition 2.14 of [38] says that if D is a PVMD and S a multiplicative set of D,then $f_S : Cl(D) \to Cl(D_S)$ defined by $[I] \mapsto [ID_S]$ is surjective. Driss Nour-El-Abidine noted that neither being a PVMD nor being a Mori domain are essential to Nagata's class group theorem. What is really at work is the property that Driss dubbed as P^* : For every nonzero finitely generated ideal I the inverse I^{-1} is a v-ideal of finite type. In Theorem 1 of [21] the author states:

Theorem 36 *Suppose that D is an integral domain satisfying P^*. If S is a multiplicative set generated by prime elements of D, then $Cl(D) \cong Cl(D_S)$.*

This was taken up in [9] in a slightly different way. We called a nonzero element $r \in D$ an extractor if $rD \cap xD$ is principal for all $x \in D \backslash \{0\}$ (equivalently if $(r, x)_v$ is principal for all $x \in D \backslash \{0\}$.) Using the fact that $r \in D \backslash \{0\}$ is an extractor if and only if r belonging to a v-ideal I makes I principal (Theorem 4.1 of [9]), the following result was proved in in [9].

Theorem 37 *(Theorem 4.2 [9]). Suppose that D is an integral domain and S is a multiplicative set of D generated by extractors of D. Then the natural map $Cl_t(D) \to Cl_t(D_S)$ is injective. If D also satisfies P^* then $Cl_t(D) \to Cl_t(D_S)$ is surjective and hence an isomorphism. ****

Noting the fact that in a domain satisfying P^* a primal element is actually an extractor, [9]Corollary 3.2, one can state the following result.

Theorem 38 *Suppose that D is an integral domain and S is a multiplicative set of D generated by completely primal elements of D. If D satisfies P^* then $Cl_t(D) \to Cl_t(D_S)$ is surjective and hence an isomorphism.*

Things became interesting with the introduction of t-linked extensions. Theorem 2.2 of [3] goes as:

Theorem 39 *Let $A \subseteq B$ be a pair of integral domains with B t-linked over A. Then the map $\theta : Inv_*(A) \to Inv_*(B)$, given by $\theta(I) = (IB)_t$ is a homomorphism. Furthermore, if xA is a principal fractional ideal of A then $\theta(xA) = xB$; thus θ induces a homomorphism $\overline{\theta} : Cl_t(A) \to Cl_t(B)$, where $\overline{\theta}[I]) = [(IB)_t]$.*

Let D be a domain, and let Δ be a set of prime t-ideals of D. Then the ring $R = \cap_{P \in \Delta} D_P$ is called a subintersection of D in section 5 of [36], where it was shown that a subintersection of a PVMD is a PVMD . (This was, as noted in section 5 of [36], in analogy with " Every subintersection of a Krull domain is a Krull domain", [29].) I have brought in this terminology to facilitate the reading of the following theorem.

Theorem 40 *Let A be a PVMD, and let $B = \cap_{P \in \Delta} A_P$, be a subintersection of A, where Δ is a subset of the set of t-primes of A. Then the map $\theta : Inv_*(A) \to Inv_*(B)$, given by $\theta(I) = (IB)_t$ is a surjective homomorphism. The induced map $\overline{\theta} : Cl_t(A) \to Cl_t(B)$ is also a surjective homomorphism.*

This is the latest form of Nagata's Theorem for PVMDs, you may want to compare it with Theorem 7.1 of [29]. I have included this result to indicate that some times, estimating is a better choice than actual computing.

One way of simplifying the computing of the t-class group of a domain D is to look for a splitting or an lcm-splitting set. Recall that a saturated multiplicative set S in D is said to be a splitting set of D if for each $d \in D\backslash\{0\}$ we can write $d = rs$ where $s \in S$ and $r \in D$ is such that $rD \cap tD = rtD$ for all $t \in S$. Also for a saturated multiplicative set S, let $T = \{t \in D | tD \cap sD = tsD$ for all $s \in D\}$ and call it a multiplicative complement (or m-complement) of S. Sometimes T is denoted by $S^{\perp}$. If S is a splitting set such that $sD \cap xD$ is principal for all $x \in D$ we call S an lcm splitting set of D. Theorem 2.2 of [2] discusses properties of splitting sets in detail. We record the following for our purposes.

Theorem 41 *Let S be a saturated multiplicative set in D. Then the following are equivalent.*

(1) S is a splitting set of D,

(2) If I is an integral principal ideal of D_S then $I \cap D$ is a principal ideal of D.

From the lcm-splitting angle a splitting set S is an lcm-splitting set if and only if $D_{S^{\perp}}$ is a GCD domain. The following is a result that may give you the clue to how the splitting sets work.

Lemma 42 *Let S be a splitting multiplicatively closed subset of D with T the m-complement for S. Let $s_1, \ldots, s_n, \in S$ and $t_1, \ldots, t_n, \in T$. Then $(s_1t_1, .., s_nt_n)_v = ((s_1, ..., s_n)(t_1, ..., t_n))_v$.*

Consequently we have the following result.

Theorem 43 *Let S be a splitting multiplicatively closed subset of D with T the m-complement for S. Let $A = (\{a_\alpha\})$ (each $a_\alpha \neq 0$) be an integral ideal of D. For each α, write $a_\alpha, = s_\alpha t_\alpha$, where $s_\alpha \in S$ and $t_\alpha \in T$. Then $A_t = (((s_\alpha\})(\{t_\alpha\}))_t$. In particular, $A_t = ((S_1)(T_1))_t$, where $S_1 = \{s \in S \mid st \in A$ for some $t \in T\}$ and $T_1 = \{t \in T | st \in A$ for some $s \in S$).*

This leads to the following result.

Theorem 44 *Let S be a splitting multiplicatively closed subset of D and let T be the m-complement for S. Let $A = (\{a_\alpha\})(a_\alpha \neq 0)$ be an integral ideal of D. For each α, let $a_\alpha = s_\alpha t_\alpha$, where $s_\alpha \in S$ and $t_\alpha \in T$. Then $(AD_S)_t \cap D = (\{t_\alpha))_t$. In particular, if A is generated by elements of T, then $(AD_S)_t \cap D = A_t$ and hence $(AD_S)_t = A_tD_S$.*

Using these and some other observations, we come to the conclusion drawn in the following result..

Theorem 45 *Let D be an integral domain, S a splitting subset for D, and T the m-complement for S. Then the map $\theta : Inv_t(D) \to Inv_t(D_S)x_cInv_t(D_T)$, given by $\theta(A) = (AD_S, AD_T)$, is a monoid order isomorphism. (Heree $Inv_t(D_S)x_cInv_t(D_T)$ represents a cardinal product, a direct product in which order is defined coordinatewise.) Moreover, for $A \in Inv_t(D)$, A is integral (respectively; principal, of finite type, t-invertible) if and only if both AD, and AD, are integral (respectively;*

principal, of finite type, t-invertible).

This leads directly to the conclusion.

Corollary 46 *With the notation of Theorem 45, the map $\theta : Cl_t(D) \to Cl_t(D_S)xCl_t(D_T)$, given by $\overline{\theta}[A]) = ([AD_S], [AD])$, is a group isomorphism. In particular, the natural map $Cl_t(D) \to Cl_t(D_S)$ is a group epimorphism and is an isomorphism if and only if $Cl_t(D_T) = 0$.*

Thus, in a nutshell if we can find a a splitting set S and its m-complement T in D, then you can split the class group of a domain as a direct sum of two class groups, which may be easier to handle. Thus for example if L is an extension of $qf(D) = K$ and X an indeterminate over L, and $R = D + XL[X]$, then $Cl_t(R) \cong Cl_t(D)$ because $S = \{u + Xl(X)|$ where u is a unit and $Xl(X) \in XL[X]\}$ is a saturated multiplicative set generated by height one primes and hence is an lcm-splitting set, with m-complement $T = D\backslash\{0\}$. This gives $Cl_t(R) \triangleq Cl_t(R_S) \times Cl_t(R_T) = Cl_t(D) \times Cl_t(K + XL[X])$ and there are several ways of proving that $Cl_t(K + XL[X])$ is trivial.

A somewhat interesting modification of the above idea has been employed by Anderson and Chang in [13].

When it comes to finding class groups of somewhat involved constructions, it is hard to pass up a recent innovation by Chang. Fossum used the following construction to prove Claborn's theorem that every abelian group is the ideal class group of a Dedekind domain. Let Λ be a nonzero index set, $\{x_i, y_i, u_i, v_i | i \in \Lambda\}$ be an algebraically independent set over $D, v_i = y_iu_i/x_i$ for all $i \in \Lambda$, $Z^{(\Lambda)}$ be the direct sum of Λ-copies of the additive group of integers, and $R = D[\{x_i, y_i, u_i, v_i | i \in \Lambda\}]$.According to Corollary 14.9 of [29], if D is a Krull domain, then so is R and $Cl(R) = Cl(D) \bigoplus Z^{(\Lambda)}$. In [24], Chang adapts this construction to prove that every Abelian group is the ideal class group of a Prufer domain of finite character that is not a Dedekind domain. Of course as Fossum [29] moves from the ideal class group of a Dedekind domain to the divisor class group of a Krull domain Chang [24] takes us from the ideal class group of a non-Dedekind Prufer ring of finite character to the t-class group of a non-Krull PVMD of finite t-character, i.e. a ring of Krull type. The whole affair is somewhat intricate, so I would leave it with a mention of [24] as a source for another computation of a t-class group.

Finally, I postponed the posting of an earlier version of the answer, hoping to write a better response and then due to my health, I forgot it completely. If you find this write up useful, thank Hwankoo Kim who pointed out to me that there was no response to the question. As most of this material got written recently and typing has been hard due to pain in my shoulders and neck. So, some errors are inevitable. I would be grateful if you point them out to me.

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