QUESTION (HD0301) If $R = F[X^3, X^4], I = (X^6, X^7, X^8)$ then how is $I^{-1} = F[X]$? (Here F is a field)

The answer partly depends upon the fact that if r and s are relatively prime positive integers then each $n \geq (r-1)(s-1)$ can be expressed as $n = xr + ys$ where x and y are nonnegative integers. Taking $r = 3$ and $s = 4$ we find that for each $n \geq (3-1)(4-1) = 6$ we can write $n = 3x + 4y$ where x and y are nonnegative integers. Thus for every $n \geq 6$ we can write $X^n = (X^3)^x(X^4)^y$. This gives us two things:

(a). For each $g \in F[X],\ gI \subseteq R.$ Reason: Let $g(X) = \sum_{i=0}^{i=m} g_i X^i$ where $g_i \in F.$ Then for $n = 6, 7, 8, \ldots,\ X^n \sum_{i=0}^{i=m} g_i X^i = \sum_{i=0}^{i=m} g_i X^{i+n} \in R$ because each of X^{i+n} is a product of powers of X^3 and X^4. So $F[X] \subseteq I^{-1}$.

(b). Every $f(X) \in R = F[X^3, X^4]$ can be written as $f(X) = f_0 + f_1X^3 + f_2X^4 + X^6 f_3(X)$ where $f_0, f_1, f_2 \in F$ and $f_3(X) \in F[X]$.

We have seen from part (a) that $F[X]I \subseteq R$ and so $F[X] \subseteq I^{-1}.$ Now we show that $I^{-1} \subseteq F[X].$ For this we note that
$I^{-1} = (X^6, X^7, X^8)^{-1} = \frac{R}{X^6} \cap \frac{R}{X^7} \cap \frac{R}{X^8} = \frac{1}{X^{21}}(X^{15}R \cap X^{14}R \cap X^{13}R) = \frac{X^7}{X^{21}}(X^8R \cap X^7R \cap X^6R) = \frac{1}{X^{14}}$
Now if we can show that each $f \in (X^8R \cap X^7R \cap X^6R)$ is of the form $X^{14}g(X)$ where $g \in F[X]$ then we are done. Now $f \in X^8R, X^7R, X^6R$ implies that $f = X^8\varphi = X^7\psi = X^6\omega$ where $\varphi, \psi, \omega \in R$. So, using (b) above, we have the following picture:

$$\begin{aligned} f = X^8\varphi &= X^8(\varphi_0 + \varphi_1X^3 + \varphi_2X^4 + X^6\varphi_3(X)) \\ &= \varphi_0X^8 + \varphi_1X^{11} + \varphi_2X^{12} + X^{14}\varphi_3(X) \qquad (i) \\ f = X^7\psi &= X^7(\psi_0 + \psi_1X^3 + \psi_2X^4 + X^6\psi_3(X)) \\ &= \psi_0X^7 + \psi_1X^{10} + \psi_2X^{11} + X^{13}\psi_3(X) \qquad (ii) \\ f = X^6\omega &= X^6(\omega_0 + \omega_1X^3 + \omega_2X^4 + X^6\omega_3(X)) \\ &= \omega_0X^6 + \omega_1X^9 + \omega_2X^{10} + X^{12}\omega_3(X) \qquad (iii) \end{aligned}$$

Note that these are three representations of the same polynomial (in R and hence in $F[X]$). So if there is a power of X that appears in one representation but not in another then the coefficient of that power must be 0. For instance $\varphi_2 = 0$ because X^{12} does not appear in the representation (ii) of f. Proceeding this way we find that
$f = X^{14}\varphi_3(X) = X^{13}\psi_3(X) = X^{12}\omega_3(X)$. Using the fact that these equations are in $F[X]$ we conclude that $f = X^{14}\varphi_3(X)$ where $\varphi_3(X) \in F[X]$.

To sum up, we have proved that $f \in (X^8R \cap X^7R \cap X^6R)$ implies that $f \in X^{14}F[X]$. So, $(X^8R \cap X^7R \cap X^6R) \subseteq X^{14}F[X]$ which leads to $I^{-1} = \frac{1}{X^{14}}(X^8R \cap X^7R \cap X^6R) \subseteq F[X]$. This completes the proof of the fact that $I^{-1} = F[X]$.

(There could be other shorter solutions, but I could only come up with this. If any of the readers has one do please contribute.)