QUESTIONS (HD0302). How is any prime ideal minimal over a t-ideal a prime t-ideal? How can you show that a maximal t-ideal is prime? How is a maximal height-one prime ideal a prime t-ideal?

The answers depend upon the following theorem that uses Zorn's Lemma.

Theorem. Let A be an integral t-ideal of D with $A \neq D$ and let S be a multiplicative subset of D such that $A \cap S = \phi$. Then there exists a prime t-ideal $Q \subsetneq D$ that contains A such that Q is maximal with respect to the property that $Q \cap S = \phi$.

Proof. Let T be the family of all integral t-ideals B_α such that $B_\alpha \supseteq A$ and $B_\alpha \cap S = \phi$. Then T is non empty because $A \in T$. We note that since T is a family of sets it can be partially ordered under inclusion. Let $\{B_\alpha\}_{\alpha \in J}$ be a chain in T. Claim: $\bigcup_{\alpha \in J} B_\alpha$ is a t-ideal. That $\bigcup_{\alpha \in J} B_\alpha$ is an ideal is easy to see and is standard. To show that $\bigcup_{\alpha \in J} B_\alpha$ is a t-ideal all we need show is that if F is a finitely generated ideal contained in $\bigcup_{\alpha \in J} B_\alpha$. Then $F_t \subseteq \bigcup_{\alpha \in J} B_\alpha$. Using the facts that $F = (a_1, a_{2,} \ldots a_r)$ is finitely generated and that B_α are in a chain we conclude that $F \subseteq B_k$ for some $k \in J$. But then $F_t \subseteq (B_k)_t = B_k \subseteq \bigcup_{\alpha \in J} B_\alpha$. Indeed $(\bigcup_{\alpha \in J} B_\alpha) \cap S = \phi$, because $(\bigcup_{\alpha \in J} B_\alpha) \cap S = \bigcup_{\alpha \in J}(B_\alpha \cap S)$ and $B_\alpha \cap S = \phi$ for each α.

To sum up we have shown that T is a non-empty partially ordered set (under inclusion) such that T contains the union of every chain in it. But then by Zorn's Lemma T contains a maximal element Q. (This means that Q is a t-ideal maximal with respect to being disjoint from S and that Q contains A.) Now we show that Q is indeed a prime ideal. Let $x, y \in D$ such that $xy \in Q$ and suppose by way of getting a contradiction that $x \notin Q$ and $y \notin Q$. Then as Q is maximal with respect to being disjoint from S we have $(x, Q)_t \cap S \neq \phi$ and $(y, Q)_t \cap S \neq \phi$ while $((x,Q)_t(y,Q)_t)_t = ((x,Q)(y,Q))_t = (xy, xQ, yQ, Q^2)_t \subseteq Q$ a contradiction.

Corollary 1. Let A be a t-ideal with $A \subsetneq D$. Then every prime ideal minimal over A is a t-ideal.

Proof. Let P be a minimal prime ideal of A and let $S = D \backslash P$. Then S is a multiplicative set and by the above theorem there is a prime t-ideal Q that contains A and that is maximal w.r.t. being disjoint from $S = D \backslash P$. But then $Q \subseteq P$. But Q cannot be properly contained in P because P is a minimal prime of A. Hence $Q = P$ making P a prime t-ideal.

Corollary 2. Let A be a t-ideal with $A \subsetneq D$. Then there is a maximal t-ideal Q containing A. Moreover Q is prime.

Proof. Take $S = \{1\}$.

Corollary 1 contains the answer to your question. There is a shorter answer given by Hedstrom and Houston [J. Pure. Appl. Algebra 18(1980) 37-44]. Their proof goes as follows: Let P be a minimal prime of a t-ideal $A \subsetneq D$. The PD_P is minimal over AD_P and so $PD_P = rad\ (AD_P)$. To show that P is a t-ideal we show that for each nonzero finitely generated ideal $F \subseteq P$, we have $F_t \subseteq P$. So let F be a finitely generated nonzero ideal contained in P. The $FD_P \subseteq PD_P = rad(AD_P)$ which means that there is a natural number n such that $(FD_P)^n = F^n D_P \subseteq AD_P$. But then there is $s \in D \backslash P$ such that $sF^n \subseteq A$. Since A is a t-ideal $(sF^n)_t \subseteq A$. Now $s(F_t)^n \subseteq (s(F_t)^n)_t \subseteq A \subseteq P$. As $s \notin P$ we have $(F_t)^n \subseteq P$ and this leads to $F_t \subseteq P$.

I like this second proof also but it seems to take you away from the mainstream star operations.

The last question can be answered as follows:

Every height 1 prime is a minimal prime of a principal (nonzero) ideal and principal ideals are all t-ideals. So every height one prime is a t-ideal (by Corollary 1 above) and if a height one prime ideal happens to be a maximal ideal then it is a maximal t-ideal. (The thing is, a maximal t-ideal may still be (properly) contained in a maximal ideal, but when a maximal ideal is contained in some (proper ideal) it must be equal to that ideal. For example let D be a local integrally closed Noetherian domain with maximal ideal M such that $ht(M) > 1$. Then, being integrally closed Noetherian, D is a Krull domain. (The famous Mori-Nagata theorem.) But, in a Krull domain every maximal t-ideal is of height one, so each height one prime is a maximal t-ideal and each of these maximal t-ideals is properly contained in the maximal ideal M in the above example. Read it carefully and understand.) Recall that an integral domain D is a Krull domain if for each height one prime P of D the localization D_P is a discrete rank one valuation domain and every nonzero nonunit of D belongs to at most a finite number of height one primes of D.