QUESTION (HD0305): If D is a PVMD and X an indeterminate over D then how can we show that $D[X]$is a PVMD?

Crucial to the understanding of the answer to this question is a command of the theory of rings of fractions or at least a working knowledge of the following results If you know localization etc. then skip the detail and get down to the answer.:

(i). If S is a multiplicatively closed set of D and P is a prime ideal of D such that $P \cap S = \phi$ then $D_P = (D_S)_{P_S}$ (P_S denotes PD_S). You can find the result in Gilmer's book on Multiplicative ideal theory [Marcel Dekker, 1972, page 54 (Cor. 5.3)]

(ii) If D is a PID and P a prime ideal of P then D_P is a (discrete) valuation domain. (If you do not know then do this: Note that a PID is a Prufer domain (Prufer means every finitely generated ideal is invertible, and every principal ideal is invertible). Now read Theorem 64 of [Kaplansky, Commutative Rings, Allyn and Bacon, 1970]. (The theorem can be stated as: An integral domain R is Prufer $\Leftrightarrow$ R_P is a valuation domain for each prime ideal P of R $\Leftrightarrow$ R_M is a valuation domain for every maximal ideal M of R, but if you did not know the theorem, you must read the proof. For the discrete part note that a PID is Noetherian, so for each prime ideal P, D_P is Noetherian and a Noetherian valuation ring can be easily shown to be a discrete rank one valuation domain.

(iii). If S is a multiplicatively closed set of an integral domain D and X an indeterminate over D then $D[X]_S = D_S[X]$. (Standard result can be proved using the definition of the ring of fractions.)

There are several ways of proving that if D is a PVMD and X an indeterminate over D then $D[X]$ is a PVMD. Of these the easiest in my opinion is via maximal t-ideals. For this we need to collect some more facts.

(iv) We know that D is a PVMD if and only if for each maximal t-ideal P of D we have that D_P is a valuation domain. ([Griffin, Some results on v-multiplication rings" Canad. J. Math.19(1967) 710-722).

(v) We know also that if M is a maximal t-ideal of $D[X]$ with $M \cap D \neq 0$ then $M = (M \cap D)[X]$ where $M \cap D$ is a maximal t-ideal of D [Prop 1.1 in "On t-invertibility II" by Houston and Zafrullah, Comm. Algebra 17(8)(1989) 1955-1969].

(vi) We also know that if D is any domain, with quotient field K, and M is a prime ideal of $D[X]$ with $M \cap D = (0)$ then $(D[X])_M$ is a valuation domain. (This is because: If $M \cap D = (0)$ then $S = D\backslash\{0\}$ is a multiplicatively closed set, and $S \cap M = \phi$. So $D[X]_M = (D[X]_S)_{M_S}$ (By (i) above).Now using (iii) $D[X]_M = (D[X]_S)_{M_S} = (D_S[X])_{M_S} = (K[X])_{M_S}$ a localization of the PID $K[X]$ and hence is a valuation domain.

ANSWER: We show that if D is a PVMD then for every maximal t-ideal M of $D[X]$, $D[X]_M$ is a valuation domain.

Let M be a maximal t-ideal of $D[X]$. If $M \cap D = (0)$ then we know that $(D[X])_M$ is a valuation domain (by (vi) above). So let M be a maximal t-ideal of $D[X]$ such that $M \cap D = P \neq (0)$. Let $S = D\backslash M = D\backslash P$ and note that $D_S = D_P$ is a valuation domain, because

P is a maximal t-ideal of the PVMD D. Then $(D[X])_M = (D[X]_S)_{MD[x]_S} = (D_S[X])_{MD_S[X]} = (D_P[X])_{MD_P[X]} = (D_P[X])_{(PD_P[X])} = D_P(X)$ a trivial extension of the valuation domain D_P. (Note: If you need to know how the trivial extension of a valuation domain is again a valuation domain look up the answer to HD0304.)