QUESTION (HD0306): If D is a Prufer v-multiplication domain and if Q is a prime t-ideal of D then how is Q[X] a prime t-ideal of D[X]?

Answer. We go a bit general and state as an anwer the following:

If Q is a prime ideal in a domain D such that D_Q is a valuation domain then $Q[X]$ is a prime t-ideal.

To understand this answer you need to know the following:

(1) If S is a multiplicative set disjoint from a prime ideal P then $D_P = (D_S)_{P_S}$. You can find the result in Gilmer's book on Multiplicative ideal theory [Marcel Dekker, 1972, page 54 (Cor. 5.3)]

(2) if V is a valuation domain with maximal ideal M then $(V[X])_{(M[X])}$ is a valuation domain. (See HD0304)

(3) if Q is a prime ideal of D such that QD_Q is a prime t-ideal of D_Q then Q is a prime t-ideal of $D.$ (You may need a proof of (3): Let $A \subseteq Q$ be a finitely generated nonzero ideal then $(AD_Q)_v = (A_vD_Q)_v$ [Zafrullah, Finite conductor domains" Manuscripta Math. 24(1978) 191-203]. Now since QD_Q is a prime t-ideal $(AD_Q)_v \subseteq QD_Q$. But since $(AD_Q)_v = (A_vD_Q)_v$, we have $A_vD_Q \subseteq QD_Q$ which forces $A_v \subseteq Q$.)

Now for the answer. Since Q is a prime t-ideal of a PVMD D, D_Q is a valuation domain. Now consider $(D[X])_{Q[X]}$ and let $S = D \backslash Q$. Then by (1) above $(D[X])_{Q[X]} = ((D[X])_S)_{Q[X]_S} = D_S[X]_{(Q_S[X])} = D_Q[X]_{(QD_Q[X])}$. But as D_Q is a valuation domain, by (2) $D_Q[X]_{(QD_Q[X])}$ is a valuation domain and so is $(D[X])_{Q[X]}$. So $Q[X](D[X])_{Q[X]}$ is a prime t-ideal and (3) applies to force $Q[X]$ to be a t-ideal.