QUESTION (HD0307): Let $S = \{X^\alpha : \alpha \in Q^+\}$ where Q^+ denotes the set of nonnegative rational numbers. Let R be the semi-group ring $Q[S]$. If $I = (X-1)R$ is Ia radical ideal?

For the answer note that R can be regarded as an ascending union of the polynomial rings $R_{n!} = Q[X^{\frac{1}{n!}}]$ where $n!$ denotes the factorial of the natural number n. That is $R = \bigcup R_{n!}$, where obviously, $R_1 \subset R_{2!} \subset R_{3!} \subset .. \subset R_{n!} \subset R_{(n+1)!} \subset \ldots$ (Note: $Q[Q^+]$ can also be viewed as a directed union of $Q[X^{\frac{1}{n}}]$ where n varies over natural numbers.)

In view of this observation, if we show that $(X-1)R_{n!}$ is a radical ideal in each of $R_{n!}$ then using the theory of ascending unions we can show that $(X-1)R$ is a radical ideal. Now to show this we note that in $R_{n!} = Q[X^{\frac{1}{n!}}]$, $X - 1 = (X^{\frac{1}{n!}})^{n!} - 1$. Now the following general lemma will help.

Lemma A. Let K be a field with characteristic 0 and let X be an indeterminate over K and let $D = K[X]$. Then $(X^n - 1)D$ is a radical ideal for every natural number n.

Proof. We first show if $(f(X))^m$ divides $(X^n - 1)$ then $m = 1$. To see this suppose that $(X^n - 1) = (f(X))^m g(X)$. (Then clearly, $f(0) \neq 0 \neq g(0)$. Differentiating both sides, with respect to X, we get $nX^{n-1} = m(f(X))^{m-1} f'(X)g(X) + (f(X))^m g'(X)$ which forces $(f(X))^{m-1}$ to divide nX^{n-1}. But this is possible only if $(f(X))^{m-1}$ is a unit, which means that $m = 1$. From this it follows that $(X^n - 1)$ is a product of distinct (mutually non associated) primes of $K[X]$. But then as distinct nonassociated primes of $K[X]$ are maximal ideals of height 1 we decide that $(X^n - 1)D$ is an intersection of principal primes. This establishes $(X^n - 1)D$ as a radical ideal.

Note here that if K were algebraically closed then all those nonassociated primes would be linear polynomials.

Now suppose that there is an $f \in R = Q[Q^+]$ such that for some m we have $(f(X))^m \in (X-1)R$. Since R is an ascending union of $\{R_{n!}\}$, $f(X)$ is a polynomial in $R_{k!}$ for some k. But in $R_{k!}$, $((X^{\frac{1}{k!}})^{k!} - 1)R_{k!}$ is a radical ideal. So that in $R_{k!}$, $(X-1)$ divides $f(X)$. But then $f(X) = (X-1)h(X)$ gives $h(X)$ in $R_{k!}$ and hence in R. Thus for all $f(X) \in R$, $(f(X))^m \in (X-1)R$ implies that $f(X) \in (X-1)R$.

Remarks and comments.

1. You may regard, for any field K, $K[Q^+]$ as a directed union $\bigcup_{n \geq 1} K[X^{\frac{1}{n}}]$ also, I learned about $K[Q^+]$ being the ascending union $\bigcup_{n \geq 1} K[X^{\frac{1}{n!}}]$ as used in the answer from Robert Gilmer.

2. If you are familiar with the notion of algebraic closure of a field then you need not use Lemma A. Just note that $(X^n - 1)K[X]$ is a product of non associated primes, and hence a radical ideal, in $\tilde{K}[X]$ where $\tilde{K}$ is the algebraic closure of K.

3. (David Anderson). Lemma A shows that if K is any field of characteristic 0 and $S = Q^+$ then $(X-1)K[S]$ is a radical ideal. However, the requirement that characteristic is zero is a must. For example if $K = Z/2Z$ then $I = (X+1)K[S]$ is not a radical ideal, because $(X^{\frac{1}{2}} + 1)^2 \in I$, but $(X^{\frac{1}{2}} + 1) \notin I$.