QUESTION (HD0309): If D is an integral domain, and M a prime ideal of $D[X]$ with $M \cap D = (0)$ then how is $D[X]_M$ a valuation domain?

This question has already been answered indirectly in **HD0305**, in the context of PVMD's. For the sake of clarity we include the general answer here.

Crucial to the understanding of the answer to this question is a command of the theory of rings of fractions or at least a working knowledge of the following results. If you know localization etc. then skip the detail and get down to the answer.:

(i). If S is a multiplicatively closed set of D and P is a prime ideal of D such that $P \cap S = \phi$ then $D_P = (D_S)_{P_S}$ (P_S denotes PD_S). You can find the result in Gilmer's book on Multiplicative ideal theory [Marcel Dekker, 1972, page 54 (Cor. 5.3)]

(ii) If D is a PID and P a prime ideal of P then D_P is a (discrete) valuation domain. (If you do not know then do this: Note that a PID is a Prufer domain (Prufer means every finitely generated ideal is invertible, and every principal ideal is invertible). Now read Theorem 64 of [Kaplansky, Commutative Rings, Allyn and Bacon, 1970]. (The theorem can be stated as: An integral domain R is Prufer $\Leftrightarrow$ R_P is a valuation domain for each prime ideal P of R $\Leftrightarrow$ R_M is a valuation domain for every maximal ideal M of R, but if you did not know the theorem, you must read the proof. For the discrete part note that a PID is Noetherian, so for each prime ideal P, D_P is Noetherian and a Noetherian valuation ring can be easily shown to be a discrete rank one valuation domain.

(iii). If S is a multiplicatively closed set of an integral domain D and X an indeterminate over D then $D[X]_S = D_S[X]$. (Standard result can be proved using the definition of the ring of fractions.)

ANSWER: If D is any domain, with quotient field K, and M is a prime ideal of $D[X]$ with $M \cap D = (0)$ then $S = D\backslash\{0\}$ is a multiplicatively closed set contained in D, and $S \cap M = \phi$. So $D[X]_M = (D[X]_S)_{M_S}$ (By (i) above).Now using (iii)
$D[X]_M = (D[X]_S)_{M_S} = (D_S[X])_{M_S} = (K[X])_{M_S}$ a localization of the PID $K[X]$ and hence is a valuation domain by (ii).

NOTES: (1) Because it is so elementary the proof of (iii) may be hard to find. So here is one: We show that $D[X]_S \subseteq D_S[X]$ and $D_S[X] \subseteq D[X]_S$ a standard method of proving the equality of two sets.

$D[X]_S \subseteq D_S[X]$: Let $f \in D[X]_S = \{\frac{\alpha(X)}{\beta} : \alpha(X) \in D[X],\ \beta \in S\}$. Then $f = \frac{\sum_{i=0}^{i=n} a_i X^i}{s} = \frac{1}{s}(\sum_{i=0}^{i=n} a_i X^i) = \sum_{i=0}^{i=n} \frac{a_i}{s} X^i \in D_S[X]$. Therefore $D[X]_S \subseteq D_S[X]$. Next we show that

$D_S[X] \subseteq D[X]_S$: Let $g \in D_S[X] = \{\sum_{i=0}^{i=n} \gamma_i X^i : \gamma_i \in D_S\}$. Note that each of $\gamma_i = \frac{d_i}{s_i}$ where $d_i \in D$ and $s_i \in S$. Thus we have $g = \sum_{i=0}^{i=n} \gamma_i X^i = \sum_{i=0}^{i=n} \frac{d_i}{s_i} X^i$. Letting $s = \prod_{i=o}^{i=n} s_i$ and letting $\tilde{s}_i = \frac{s}{s_i}$ we get $g = \sum_{i=0}^{i=n} \gamma_i X^i = \sum_{i=0}^{i=n} \frac{d_i}{s_i} X^i = \frac{1}{s} \sum_{i=0}^{i=n} \frac{s d_i}{s_i} X^i = \frac{1}{s} \sum_{i=0}^{i=n} \tilde{s}_i\ d_i X^i$ which clearly belongs to $D[X]_S$. This gives us the reverse inclusion$D_S[X] \subseteq D[X]_S$.

(2). Please note that the formula $D[X]_S = D_S[X]$ works only if S is a multiplicative set contained in D.