QUESTION (HD0311): Is a prime t-ideal P, of a domain R, always a maximal t-ideal? Give an example if the answer is no.

ANSWER: The answer is “not in general". For example let R be a Bezout domain (every finitely generated ideal of R is principal and hence a v-ideal). Now if A is an ideal in any integral domain D, then $A = \bigcup\{F :. F$ is a nonzero finitely generated ideals contained in $A\}$. Further if A is an ideal in the Bezout domain R then $A_t = \bigcup\{F_v : F$ is a nonzero finitely generated ideal contained in $A\}$. But since each of these finitely generated F's is principal and so is a v-ideal we have $F_v = F$. So, $A_t = \bigcup\{F_v : F$ is a nonzero finitely generated ideal contained in $A\} = \bigcup\{F :. F$ is a nonzero finitely generated ideals contained in $A\} = A$. We have established the following statement.

OBSERVATION: In a Bezout domain R, every nonzero ideal is a a t-ideal.

Now all that remains to show is that in a Bezout domain there can be prime ideals that are not maximal. For this let us take the easiest example: A valuation domain is obviously a Bezout domain. Let us take a valuation domain V of rank 2. This means that the maximal ideal M of V contains a nonzero prime ideal P. This prime ideal P is a t-ideal by the above observation, but it is not a maximal t-ideal, because it is contained in Mwhich is also a t-ideal by the above observation. (A good easy to see example of a rank 2 valuation domain is $V = Z_{(2)} + XQ[[X]] \ = \{f : f$ is a power series over Q with constant term in $Z_{(2)}\})$

NOTES: (i). I have taken a very simple example, to prove the point that not every prime t-ideal is a maximal t-ideal. There are examples of non Bezout domains with prime t-ideals that are not maximal t-ideals.

(ii). This question was asked by someone who happens to know a bit about the star operations. For someone wanting to gain a working knowledge, without having to fret about what the author is really saying, the best source for star operations is sections 32 and 34 of [Robert Gilmer, Multiplicative Ideal Theory, Marcel-Dekker, 1972]. The rest of the book is also a marvel of direct and unconvoluted treatment of the subject. To make sure that the reader who chanced at this particular question, and is now curious, here is a brief introduction to the notion of star operations.

A brief introduction to star operations

The main source of the following introduction is [Gilmer, Multiplicative Ideal Theory, Marcel-Dekker, 1972, sections 32 and 34]

Let D be an integral domain with quotient field K. A D-submodule A of K is said to be a fractional ideal of D if for some nonzero $d \in D$, we have $dA \subseteq D$. Let $F(D)$ be the set of (all) nonzero *fractional ideals* of D, and let $f(D)$ be the set of nonzero finitely generated fractional ideals of D.

Examples: (1). Let $D = Z$ (the ring of integers) then all nonzero fractional ideals of Z are Z-submodules of Q (the quotient field of Z) of the type $\frac{x}{y}Z$, where $x,y \in Z\backslash\{0\}$.

(2). Let $D = F[X,Y]$, where F is a field and X and Y indeterminates. Then $\frac{(x^2,y)}{xy}$ is a fractional ideal.

(3). A fractional ideal of D can be written as $\frac{A}{d}$, where A is an ideal of D and d is a nonzero element of D. So, every ideal of D is a fractionary ideal. The ideals of D are often called the integral ideals of D.

(4). Not all D-submodules of K are fractional ideals. For example for a multiplicative set S of D the ring of fractions D_S is a D-submodule of K that is not a fractional ideal, unless S is generated by units of D. On the other hand a fractional ideal can be an overring (ring between D and K). For example, you can verify that given the ring $D = Z + XQ[X]$, The ring $Q[X]$ is a fractional ideal. Indeed $Q[X]$ is a D-submodule of $Q(X)$.

A star operation is a function $A \mapsto A^*$ on $F(D)$ with the following properties:

If $A, B \in F(D)$ and $a \in K \backslash \{0\}$, then

(i) $(a)^* = (a)$ and $(aA)^* = aA^*$.

(ii) $A \subseteq A^*$ and if $A \subseteq B$, then $A^* \subseteq B^*$.

(iii) $(A^*)^* = A^*$.

We shall call A^* the $*$-*image* (or $*$-*envelope*) of A. An ideal A is said to be a $*$-*ideal* if $A^* = A$. Thus A^* is a $*$-ideal (by (iii)). Moreover (by (i)) every principal fractional ideal, including $D = (1)$, is a $*$- ideal for any star operation $*$.

For all $A, B \in F(D)$ and for each star operation $*$, $(AB)^* = (A^*B)^* = (A^*B^*)^*$. These equations define what is called $*$-*multiplication (or* $*$-*product*).

If $\{A_\alpha\}$ is a subset of $F(D)$ such that $\cap A_\alpha \neq (0)$ then $\cap (A_\alpha)^* = (\cap (A_\alpha)^*)^*$. We may call this property the *intersection property*.

Also if $\{A_\alpha\}$ is a subset of $F(D)$ such that $\sum A_\alpha$ *is a fractional ideal*, then $(\sum A_\alpha)^* = (\sum A_\alpha^*)^*$; this may be called the sum property.

Note that by the intersection property any nontrivial intersection of $*$- ideals is again a $*$-ideal. Thus for $a, b \neq 0$, $(a) : b = \{x \in D | xb \subseteq (a)\}$ is a $*$-ideal for any star operation $*$ because $(a) : b = (\frac{a}{b}) \cap D$ and so, for each $A \in F(D)$, is $A^{-1} = \{x \in K | xA \subseteq D\}$ $(= \cap_{a \in A} (\frac{1}{a}) = D :_K A$. On the particular side, if $A \in F(D)$ is a $*$-ideal for some star operation $*$ and $B \in F(D)$ then so is $A :_K B = \bigcap_{b \in B - \{0\}} \frac{A}{b}$ (by the intersection property).

Define $A_v = (A^{-1})^{-1}$ and $A_t = \bigcup \{F_v | \, 0 \neq F$ is a finitely generated subideal of $A\}$. The functions $A \mapsto A_v$ and $A \mapsto A_t$ on $F(D)$ are more familiar examples of star operations defined on an integral domain. A v-ideal is better known as a *divisorial* ideal. The identity function d on $F(D)$, defined by $A \mapsto A$ is another example of a star operation. There are of course many more star operations that can be defined on an integral domain D. But for any star operation $*$ and for any $A \in F(D)$, $A^* \subseteq A_v$. Some other useful relations are: For any $A \in F(D)$, $(A^{-1})^* = A^{-1} = (A^*)^{-1}$ and so, $(A_v)^* = A_v = (A^*)_v$.

One classical method of getting star operations is via defining families of overrings of D. Here, by a defining family we mean a family $\{R_\alpha\}$ of overrings of D such that $D = \bigcap_\alpha R_\alpha$. Now if $\{R_\alpha\}$ is a defining family of overrings of D the function $A \mapsto A^* = \bigcap_\alpha AR_\alpha$, on $F(D)$ is a star operation, which is said to be *induced* by $\{R_\alpha\}$. Indeed if $*$ is induced by $\{R_\alpha\}$, then $A^*R_\alpha = AR_\alpha$. Traditionally, the defining families consisted of valuation domains. These days star operations induced by quotient rings, preferably localizations at primes, are in vogue. An interested reader may want to see the following paper and references there [D.D. Anderson,"Star operations induced by overrings" Comm. Algebra 16(1988) 2535-2553].