QUESTION (HD0312): If P is a prime t-ideal of an integral domain R, must PR_P be a prime t-ideal of R_P?

ANSWER: **Not generally**. The argument used in the explanation will involve a good understanding of the following results:

(1) If S is a multiplicatively closed set of D and P is a prime ideal of D such that $P \cap S = \phi$ then $D_P = (D_S)_{P_S}$ (P_S denotes PD_S). You can find the result in Gilmer's book on Multiplicative ideal theory [Marcel Dekker, 1972, page 54 (Cor. 5.3)].

(2) If QD_Q is a prime t-ideal of D_Q then Q is a prime t-ideal of D (See HD0306, for a proof of this statement)

(3) An integral domain D is a PVMD if and only if D_P is a valuation domain for every maximal t-ideal P of R (see Corollary 4.3 of [Mott and Zafrullah, On Prufer v-multiplication domains, Manuscripta Math. 35(1981) 1-26]).

(4) A domain D is a GCD domain if and only if for every nonzero finitely generated ideal A of D we have A_t principal and hence t-invertible. So, a GCD domain is a PVMD (for every finitely generated nonzero A, A_t is t-invertible).

Let us establish the answer indirectly. Let us note that there exist domains R that are locally PVMD (i.e. for each maximal ideal M of R we have that R_M is a PVMD) but R is not a PVMD. One such example is Example 2.6 of [Zafrullah, The $D+XD_S[X]$ construction from GCD domains, Journal of Pure Appl. Algebra, 50(1988) 93-107]. It is given as an example of a so called P-domain, that is not a PVMD but as pointed out on page 104 of the same article this Example 2.6 is that of a locally GCD domain . Once you have convinced yourself that there is a locally GCD domain R that is not a PVMD argue as follows:

Suppose on the contrary that for every prime t-ideal P we have PR_P a t-ideal for, every domain and hence for, this domain R. Let P be any maximal t-ideal of R. Then P is contained in a maximal ideal M of R and $R_P = (R_M)_{PR_M}$ (by (1)) so that $PR_P = P(R_M)_{PR_M} = PR_M(R_M)_{PR_M}$. Now if PR_P is a prime t-ideal then so is $PR_M(R_M)_{PR_M}$and so is $PR_M(R_M)_{PR_M} \cap R_M = PR_M$ a prime t-ideal of R_M (by (2)). But then PR_M is a prime t-ideal of the GCD domain R_M and so $(R_M)_{PR_M}$ is a valuation domain (by (3)). But $(R_M)_{PR_M} = R_P$ as we have seen above. Since we had assumed that P is any maximal t-ideal of R we have shown that for every maximal t-ideal P of the given (locally GCD non-PVMD) domain R, R_P is a valuation domain, forcing R to be a PVMD by (3) above. But this contradicts the fact that R is not a PVMD.

For a direct proof that involves constructing such an example look up Zafrullah [Well-behaved prime t-ideals, Journal of Pure and Applied Algebra, 65(1990) 199-207]. Some simpler examples of domains D that have prime t-ideals P such that PD_P is not a t-ideal have been constructed in a recent survey article [Zafrullah, Various Facets of rings between $D[X]$ and $K[X]$, Comm. Algebra 31(5)(2003) 2497-2540].