QUESTION:(HD0314) What is a pullback? Give some examples.

ANSWER:

The notion of a pullback comes from category theory where it shows up as a special case of the inverse limit problem. A good description of it can be found on pages 51 and 52 of J.J. Rotman’s book [An introduction to homological algebra, Academic Press, 1979]. Briefly if \( f \) and \( g \) are maps from objects \( B \) and \( C \) to object \( A \) as shown:

\[
\begin{array}{c}
\text{C} \\
\downarrow g \\
\text{B} \xrightarrow{f} \text{A}
\end{array}
\]

The pullback is an object \( L \) and maps \( \alpha, \beta \) such that the diagram

\[
\begin{array}{c}
\text{L} \\
\downarrow \alpha \\
\text{B} \xrightarrow{f} \text{A}
\end{array}
\]

\[
\begin{array}{c}
\text{C} \\
\downarrow g
\end{array}
\]

commutes and has the property that if there were another object \( M \) with maps \( u : M \to C \) and \( v : M \to B \) such that

\[
\begin{array}{c}
\text{M} \\
\downarrow v \\
\text{B} \xrightarrow{f} \text{A}
\end{array}
\]

\[
\begin{array}{c}
\text{C} \\
\downarrow g
\end{array}
\]

commutes then there is a unique map \( \theta : M \to L \) such that

\[
\begin{array}{c}
\text{M} \quad \xrightarrow{u} \quad \text{C} \\
\downarrow \theta \\
\text{L} \xrightarrow{\alpha} \text{C}
\end{array}
\]

\[
\begin{array}{c}
\text{B} \xrightarrow{f} \text{A}
\end{array}
\]

commutes.

As indicated on page 53 of Rotman’s book, if we complete

\[
\begin{array}{c}
\text{C} \\
\downarrow g \\
\text{D} \xrightarrow{\overline{\alpha}} \text{C}
\end{array}
\]

with \( \downarrow \beta \) and \( \downarrow g \) such that

\[
\begin{array}{c}
\text{B} \xrightarrow{f} \text{A}
\end{array}
\]

\[
\begin{array}{c}
\text{B} \xrightarrow{f} \text{A}
\end{array}
\]

\[
\begin{array}{c}
\text{D} = \{ (b, c) \in B \times C : f(b) = g(c), \quad \alpha : (b, c) \mapsto c \text{ and } \beta : (b, c) \mapsto b \}
\end{array}
\]

then \( D \) is a pullback. Let us call the above commutative diagram the Cartesian product construction of a pullback.

Here is another scenario: Let in the following diagram...
$g$ be injective and $f$ be surjective. To make life easier we can assume that $g$ is an inclusion. Then $L = f^{-1}(C)$ satisfies the above definition of a pullback. For we can take the map $\alpha = f|_L$ and the map $\beta$ is clearly inclusion. So, with this description the diagram:

$$
\begin{array}{c}
L \\ \downarrow \alpha \\
C \\
\downarrow \beta \\
B \\
\rightarrow ^f A
\end{array}
$$

obviously commutes. Next, let $M$ be an object with maps $u : M \to C$ and $v : M \to B$ such that

$$
\begin{array}{c}
M \\
\downarrow v \\
C \\
\downarrow g \\
B \\
\rightarrow ^f A
\end{array}
$$

define $\theta : M \to L$ by $\theta(m) = v(m)$. Indeed as $g$ is an inclusion we have for each $m \in M$, $u(m) = f(v(m))$, which forces $v(m) \in f^{-1}(C) = L$. Now $\theta$ is unique because $\beta$ is an inclusion. Thus we have the following rule from the ring theory point of view.

Proposition. If $A, B, C$ are objects in a category of $R$-modules such that such that $C \subseteq A$ and $f : B \to A$ is a surjective homomorphism, then

$L = f^{-1}(C)$ is the pullback of :

$$
\begin{array}{c}
C \\
\downarrow g \\
B \\
\rightarrow ^f A
\end{array}
$$

where $g$ is an inclusion.

Just to indicate that the above proposition does not represent the only scenario, here is another scenario. Let $B$ and $C$ be subsets of an object $A$ then $B \cap C$ is a pullback of the injective maps $B$ to $A$ and $C$ to $A$. This remark has been used by Grothendieck to define

Here are a few examples of pullbacks commonly used in the context of integral domains:

(1). Let $M$ be a maximal ideal of an integral domain $R$. Then $R/M$ is a field. Let $A$ be a subring of $R/M$. Now we have $f : R \to R/M$ the canonical surjection and $g : A \to R/M$ the injection (inclusion). In pictures these functions can be represented by

$$
\begin{array}{c}
A \\
\downarrow g \\
R \\
\rightarrow ^f R/M
\end{array}
$$

Now let $D = f^{-1}(A)$ which can be shown to be a ring. Then by the above Proposition
\[ D = f^{-1}(A) \] is a pullback

\[
\begin{array}{ccc}
D & \xrightarrow{\alpha} & A \\
\downarrow \beta & & \downarrow g \\
R & \xrightarrow{f} & R/M
\end{array}
\]

commutes, where \( \alpha \) is is surjective and \( \beta \) is injective. So \( D \) can be identified with a subring of the original ring \( R \).

Based on the same theme here are some examples: Consider for \( R \), the polynomial ring \( Q[X] \) where \( Q \) is the field of rational numbers and \( X \) is an indeterminate over \( Q \). Now \( R = Q[X] \) is a PID, so every nonzero prime ideal of \( R \) is a maximal ideal of \( R \). Take \( M = XQ[X] \). Then \( R/M \cong Q \). Choose \( Z \) the ring of integers to serve for \( A \). As a subring of \( R/M \), \( Z \cong \{z + M : z \in Z\} \). Next take the canonical surjection \( f : Q[X] \to Q[X]/XQ[X] \cong Q \) and consider \( f^{-1}(z + M : z \in Z) = \) polynomials whose images under \( f \) are in the set of cosets \( \{z + M : z \in Z\} \). Now these are precisely the polynomials in \( Q[X] \) whose constant terms are in the set of integers \( Z \). Thus

\[ f^{-1}(\{z + M : z \in Z\}) = \{h(X) \in Q[X] : h(0) \in Z\} = \{a_0 + \sum a_i X^i : a_0 \in Z, a_i \in Q\} \]

is easy to see that each \( \sum a_i X^i \in XQ[X] \). So we can write

\[ f^{-1}(\{z + M : z \in Z\}) = Z + XQ[X] \]

and it is easy to see that \( Z + XQ[X] \) is a subring of \( Q[X] \). So we have the following picture:

\[
\begin{array}{ccc}
Z + XQ[X] & \xrightarrow{\alpha} & \{z + M : z \in Z\} \\
\downarrow \beta & & \downarrow g \\
R = Q[X] & \xrightarrow{f} & \{q + M : q \in Q\}
\end{array}
\]

Now this is what happens: Take \( h(X) = a_0 + \sum a_i X^i \in Z + XQ[X] \), the map \( \alpha \) assigns to \( h(X) \) the coset \( a_0 + M \) and \( g \) being the inclusion maps this coset into \( \{q + M : q \in Q\} = R/M \) as the coset \( a_0 + M \). So, we have \( ga(a_0 + \sum a_i X^i) = a_0 + M \). Next as \( Z + XQ[X] \) is a subring of \( Q[X] \) we can take \( \beta \) to be the inclusion map. Then for \( h(X) = a_0 + \sum a_i X^i \in Z + XQ[X] \) we have

\[ \beta(a_0 + \sum a_i X^i) = a_0 + \sum a_i X^i \in Q[X] \]

and \( f \) being the canonical surjection takes \( a_0 + \sum a_i X^i \) into \( a_0 + M \). But then for all \( h(X) \) in \( Z + XQ[X] \) we have \( ga(h(X)) = f\beta(h(X)) \), which means that the diagram

\[
\begin{array}{ccc}
Z + XQ[X] & \xrightarrow{\alpha} & \{z + M : z \in Z\} \\
\downarrow \beta & & \downarrow g \\
R = Q[X] & \xrightarrow{f} & \{q + M : q \in Q\}
\end{array}
\]

commutes. Thus we see that \( Z + XQ[X] \) is a pullback of

\[ \{z + M : z \in Z\} \]

\[
\begin{array}{ccc}
\downarrow g & & \\
R = Q[X] & \xrightarrow{f} & \{q + M : q \in Q\}
\end{array}
\]

where \( M = XQ[X] \), \( \{q + M : q \in Q\} \) is the set of cosets of \( Q[X]/XQ[X] \), \( \{z + M : z \in Z\} \) is the set of cosets (which is a subring of \( Q[X]/XQ[X] \) that represent polynomials with constant
term in \(Z\), \(g\) is the inclusion map from \(\{z + M : z \in Z\}\) into the quotient \(Q[X]/XQ[X]\) and \(f\) is the canonical surjection.

Note: Now that the description part is over, we can do away with some of the details. Noting that \(\{z + M : z \in Z\} \cong Z\) and that \(Q[X]/XQ[X] \cong Q\) we can replace (II) by

\[
\begin{array}{ccc}
Z + XQ[X] & \xrightarrow{\gamma} & Z \\
\downarrow \beta & & \downarrow g \\
R = Q[X] & \xrightarrow{f} & Q \cong Q[X]/XQ[X]
\end{array}
\]

and say that \(Z + XQ[X]\) is a pullback of

\[
\begin{array}{ccc}
Z & \\
\downarrow g
\end{array}
\]

\[
R = Q[X] \xrightarrow{f} Q \cong Q[X]/XQ[X]
\]

Now looking at this example you can make your own pullbacks of this kind. For example, if \(D\) is an integral domain which is contained (as a subring) in a field \(L\) then \(D + XL[X]\) is a pullback of

\[
\begin{array}{ccc}
D & \\
\downarrow g
\end{array}
\]

\[
R = L[X] \xrightarrow{f} L \cong L[X]/XL[X]
\]

where \(g\) is the canonical injection and \(f\) is the canonical surjection.

My description has been lengthy because, I do not know how much the reader already knows. Usually, as already indicated, a pullback (domain) is described as: Let \(M\) be a maximal ideal of a domain \(R\), \(f : R \twoheadrightarrow R/M\) the canonical surjection. If \(A\) is a subring of \(R/M\) then \(f^{-1}(A)\) is a pullback of \(A\) in \(R\) over \(R/M\). This approach is responsible for a number of interesting pullback constructions:

1. The \(D + M\) construction: Let \(V = k + M\) be a valuation domain, where \(k\) is a field and \(M\) is the maximal ideal of \(V\). For each subring \(D\) of \(k\) then \(D + M\) is a pullback of \(D\) in \(V\) over \(k\). This kind of pullback constructions, being a good source of examples of a certain type, have been extensively used by Gilmer and his students and quite a few other Mathematicians. Gilmer’s book, [Multiplicative Ideal Theory, Marcel Dekker, 1972] is a good source for these constructions. Another relevant reference is R. Gilmer and E. Bastida [Michigan Math. J. 20 (1973), 79–95].

2. The generalized \(D + M\) construction: This notion was introduced by J. Brewer and E. Rutter in [Michigan Math. J. 23 (1976), no. 1, 33–42]. This construction requires a domain \(R\) expressible as \(R = k + M\) where \(k\) is a field and \(M\) is a maximal ideal of \(R\). Pick a suitable subring \(D\) of \(k\) to make the ring \(D + M = \{d + m : d \in D\) and \(m \in M\}\). The generalized \(D + M\) construction includes the \(D + XL[X]\) construction and is a pullback for similar reasons. However, it is a bit more versatile in that it also allows constructions like \(D + XL[[X]]\), the ring of power series over a field \(L\) with constant terms in a subring \(D\).

There is another, more general, approach to constructing pullbacks. It goes as follows: Let \(R\) be a domain, let \(M\) be a nonzero prime ideal of \(R\) and let \(A\) be a subring of the domain
Consider:

\[
\begin{array}{c}
A \\
\downarrow g \\
R \xrightarrow{f} R/M
\end{array}
\]

where \(g\) is injective. Then the subring \(D\) of \(R\) is a pullback if the diagram

\[
\begin{array}{c}
D \\
\downarrow \beta \\
R \xrightarrow{f} R/M
\end{array}
\]

commutes. Indeed this \(D\) turns out to be \(f^{-1}(A)\) by the Proposition above.

One of the easier constructions using this general approach is the, so called, \(A + XB[X]\) construction where \(A \subseteq B\) is an extension of domains and \(A + XB[X]\) denotes the ring of polynomials over \(B\) with constants in \(A\).

Indeed there appears to be a common thread in the above examples. Each of these examples consists of a pair of rings \(R \subseteq S\) such that \(R\) and \(S\) contain a common ideal. It turns out that in such a pair \(R \subseteq S\), \(R\) is a pullback.

I have used the simplest possible examples to explain the theory. For more theoretical treatment consult Marco Fontana’s Topologically defined classes of commutative rings, Ann. Mat. Pura Appl., IV. Ser. 123(\textbf{1980}), 331-355.

There is extensive literature on pullbacks. Yet from the Multiplicative Ideal Theory point of view the following references will be useful.


On the \(A + XB[X]\) constructions the reader may want to consult the following articles and references there:


(The following have helped in preparing this answer: David Anderson, Gabriel Picavet and Martine Picavet.)