QUESTION (HD0401): If S is a saturated multiplicative set in a domain R and if a, b are two nonzero elements of R such that $\frac{a}{b}$ belongs to R_S, must b be in S?

ANSWER: Not necessarily. For any multiplicative set S, saturated or not, in a domain R, $R_S = \{a/b|$ where $a \in R$ and $b \in S\}$. This is the usual definition. This means that every element p/q of R_S is equivalent to an element of the form a/b (where $b \in S$) under the relation $p/q = a/b$ if and only if $pb = aq$. So, for p/q to be in R_S all it needs is that there is $a \in R$ and $b \in S$ such that $p/q = a/b$. In other words we can have $p/q \in R_S$ with $q \notin S$. Here is an example: Let $R = Z$ the ring of integers and let $S = \{\pm 2^n$: where n is an integer $\geq 0\}$. Clearly S is saturated. Now $\frac{15}{12} \in Z_S$ (because $\frac{15}{12} = \frac{5}{4}$) but of course $12 \notin S$.

For a detailed treatment of rings of fractions the reader may consult sections 4 and 5 of Gilmer's book on Multiplicative ideal theory or any standard book on commutative ring theory. Note however that most of these books treat the topic for general rings and not just for integral domains.

NOTE: The reason behind this question is an erroneous statement to the effect that if $\frac{a}{b}$ belongs to R_S and if S is saturated then $b \in S$ that appears in lines 1 and 2 at page 235 of Gilmer's [Multiplicative Ideal Theory, Dekker 1972] and [Queen's papers in Pure and Applied Mathematics, Volume 90(1992)]. These lines are a part of part (2) of the proof of Theorem 19.11, but as we show below the proof can be completed in a slightly different fashion.

Starting with the last line on page 234 of Gilmer's book, the proof of part (2) of Theorem 19.11 can be completed as follows:

Thus if xU is in H then x belongs to $D_1 = D_N$ implies $x = a/b$ where $b \in N$ and $x^{-1} \in D_1 = D_N$ implies $x^{-1} = c/d$ where $d \in N$. This gives $a/b = d/c$ which forces $ac = bd$. Now since N is saturated and $b, d \in N$ we have $a, c \in N$. Thus $xU = aU - bU = cU - dU$ where aU, bU, cU, dU are positive elements of H and H is filtered.

(Comment dated: November 8, 2004. This question was asked by Muhammad Sakhdari of Kashan University (Iran). He asked the question, I gave him the negative answer. He persisted, I gave him a couple of examples and as an after-thought I asked if he had seen it somewhere. That is when the young fellow came up with the reference. Professor Gilmer was surprised that no one previously had pointed this error out to him. The trouble is people teach on the basis of what they already know and so do not look into the books critically enough; besides the error is embedded in such a way that it is hard to catch. Muhammad Zafrullah)