QUESTION:(HD0404) I wonder if there is a way to describe the (fractional) overrings of $D + XK[X]$. In particular, how can one find the (fractional) overrings of $Z + XQ[X]$? Would you be willing to suggest to me any papers or references to help me answer the above question?

ANSWER: The reference you are looking for is a paper by D. Costa, J. Mott and M. Zafrullah: Overrings and dimensions of general $D + M$ constructions, J. Natur. Sci. and Math. 26(2) (1986), 7-13.

In this paper look for Proposition 2.2 which, for your question, translates to: Each overring S of $D + XL[X]$ is a ring of fractions of $S \cap L + XL[X]$, where L is a field containing the quotient field K of D.

From this statement it follows that $S = (S \cap L + XL[X])_M$ where M consists of elements of $S \cap L + XL[X]$, which are units in S.

(The paper is written using the language of "Generalized $D + M$ construction" see Brewer and Rutter: $D + M$ construction with general overrings, Michigan Math. J. 23(1976) 33-42.)

The case for $Z + XQ[X]$ is easier since $Z + XQ[X]$ is a Bezout domain (see Costa, Mott, Zafrullah, The construction $D + XD_S[X]$, J. Algebra 53(1978) 423-439.) So every overring S of $Z + XQ[X]$ is a ring of fractions of $Z + XQ[X]$.

The paper in J. Natur. Sci and Math. has a lot of typos, but I am sure there would no major problems reading it. If however there is a problem in reading it or in acquiring a copy of it let me know.

PS. (1) It just occurred to me that you may be looking for a way of expressing an overring S of $Z + XQ[X]$ as a ring of fractions. For this proceed as follows: Let $L = \{z \in Z : z$ is a unit in $S\}$. If $L \neq Z\backslash\{0\}$ let $M = \{f = \pm 1 + Xg(X)$:where $g(X) \in Q[X]$ and f is a unit in $S\}$. Now it is easy to see that $Z_L = S \cap Q$ and $S = (Z_L + XQ[X])_M$. If $L = Z\backslash\{0\}$ then two cases arise: (a) X is not a unit in S (b) X is a unit in S. If X is not a unit in S then $S = Q[X]_M$ where M consists of all the elements of $Q[X]$ that are units in S. If X is a unit in S then S is a quotient ring of the PID $Q[X, X^{-1}]$. So $S = Q[X, X^{-1}]_M$ where M is the set of all elements of $Q[X, X^{-1}]$ which are units in S.

PS. (2). Your question is about (fractional) overrings, and this question can also be construed as a question about overrings of $D + XK[X]$ which are fractional ideals of $D + XK[X]$. (A fractional ideal of a domain D is a D-submodule F of K such that for some $d \in D\backslash\{0\}$ we have $dF \subseteq D$.) For the sake of completeness let us deal with $D + XL[X]$ where L is a field containing the quotient field of D.

Proposition. An overring S of $D + XL[X]$ is a fractional ideal of $D + XL[X]$ if and only if $S = S \cap L + XL[X]$.

Proof. ($\Rightarrow$) Suppose that $S = S \cap L + XL[X]$. Then since $XS = X((S \cap L) + XL[X]) \subseteq XL[X] \subseteq D + XL[X]$ we have the conclusion.

($\Leftarrow$) Suppose that S is an overring that is a fractional ideal then S is a quotient ring of $S \cap L + XL[X]$ and for some $h(X) \in D + XL[X]$ we have $h(X)S \subseteq D + XL[X]$. We claim that no element of the form $X^a(1 + Xg(X))$ is a unit in S, where $\alpha \geq 0$. For if that were the case there is at least one $n \in N$ such that $(X^a(1+Xg(X)))^n \nmid h(X)$ in $L[X]$ forcing $\frac{h(X)}{(X^a(1+Xg(X)))^n} \notin L[X] \supseteq D+XL[X]$. So, M consists of elements of $S \cap L$ alone, which are units in S.

(Comment: Lucian Sega of Purdue University, West Lafayette, Idiana, asked this question.)

(Salah Kabbaj helped in giving the answer a final shape.) Zafrullah