QUESTION:(HD0405) Let there be a family $\{P_\alpha; \alpha \in I\}$ of prime ideals of R such that:

(1) Each R_{P_α} is a valuation domain and $P_\alpha R_{P_\alpha}$ is divisorial

(2) the family $\{R_{P_\alpha} : \alpha \in I\}$ is a family of finite character for R

(3) each pair of $\{R_{P_\alpha} : \alpha \in I\}$ are independent.

Why for each maximal t-ideal,M, of R there is $\alpha \in I$ such that $M = P_\alpha$?

ANSWER: Let me give you a more general answer. To understand the answer you should have a working knowledge of star operations and should pay attention to the following. Recall from [Theorem 1 (6), D.D. Anderson, Star operations induced by overrings, Comm. Algebra 16(12)(1988) 2535-2553] that the star operation $*$ induced by $\{R_{P_\alpha} : \alpha \in I\}$ (of your description) is a star operation of finite character. This means that for any fractional ideal A we have $A^* = \cap AD_{P_\alpha}$ and that $A^* = \cup\{F^*$: where F ranges over nonzero finitely generated subideals of $A\}$. Now recall that $A_t = \cup\{F_v$: where F ranges over nonzero finitely generated subideals of $A\}$ and that for any star operation $*$ we have $A^* \subset A_v$. Using this we have $A^* = \cup\{F^* \subset F_v$: where F ranges over nonzero finitely generated subideals of $A\} \subseteq \cup\{F_v$: where F ranges over nonzero finitely generated subideals of $A\} = A_t$. **Thus if $*$ is a star operation of finite character then for each nonzero fractional ideal A we have $A^* \subseteq A_t$.**

Proposition. Let there be a family $\{P_\alpha; \alpha \in I\}$ of prime t-ideals of R such that $R = \cap\{R_{P_\alpha} : \alpha \in I\}$ is of finite character. If M is a maximal t-ideal of R then $M = P_\alpha$ for some $\alpha \in I$.

Proof. Let M be a maximal t-ideal and suppose that $M \nsubseteq P_\alpha$ for any α. Then $MR_{P_\alpha} = R_{P_\alpha}$ for all α and so $M^* = \cap MR_{P_\alpha} = \cap R_{P_\alpha} = R$. But since $*$ is of finite character, $R = M^* \subseteq M_t = M$ a contradiction, whence $M \subseteq P_\alpha$ for some α. Now since P_α is a prime t-ideal and M is a maximal t-ideal we conclude that $M = P_\alpha$.

Note that if for a nonzero prime ideal P we have R_P a valuation domain then P is a prime t-ideal [see HD0306]. This gives rise to the following corollary.

Corollary. Let there be a family $\{P_\alpha; \alpha \in I\}$ of prime ideals of R such that (i) R_{P_α} is a valuation domain for each $\alpha \in I$

(ii) $R = \cap\{R_{P_\alpha} : \alpha \in I\}$ is of finite character. If M is a maximal t-ideal of R then $M = P_\alpha$ for some $\alpha \in I$.

So you see that the conclusion that $M = P_\alpha$ for some α holds even in the absence of your condition (3). However, you have to make sure that (2) includes the word **defining** which means that $R = \cap R_{P_\alpha}$. If you do not, then you run into problems since there are examples of Noetherian domains R which are not integrally closed but for which R_P is a discrete rank one valuation domain for each height one prime P.

So, for such a Noetherian domain R you would have a family $\{P_\alpha\}$ of height one prime ideals such that

(1). Each R_{P_α} is a valuation domain and $P_\alpha R_{P_\alpha}$ is divisorial

(2). the family $\{R_{P_\alpha} : \alpha \in I\}$ is such that every nonzero nonunit of R is a nonunit in only a finite number of R_{P_α}

(3). each pair of members $\{R_{P_\alpha} : \alpha \in I\}$ is independent.

But since $R \neq \cap R_{P_\alpha}$, you cannot conclude that every maximal t-ideal of R is equal to some P_α.

(This question was asked by Mohammad Sakhdari)