QUESTION HD0701. Consider the following argument. Let R be a pre-Schreier domain. Then $S = R\backslash\{0\}$ is a saturated multiplicative set of completely primal elements. Now $R_S[X] = R[X]_S$ is a GCD domain and hence a Schreier domain. So by your version of Cohn's Nagata type theorem $R[X]$ is a pre-Schreier domain ([Manuscripta Math. 80(1993), Corollary 8]). But according to McAdam and Rush's work $R[X]$ pre-Schreier implies $R[X]$ Schreier. What is the reason for this discrepancy?

ANSWER. You are not the first person to have made the mistake in your "argument" and from the look of it, you would not be the last. Your mistake is in taking a completely primal element in R and assuming that it must be a completely primal element in $R[X]$. In other words, to apply my version of Cohn's Nagata type theorem, you are assuming that the set $S = R\backslash\{0\}$ consists of primal and hence completely primal elements of $R[X]$; which happens only if R is Schreier! I promise to elaborate on this point after I give some adequate introduction, so that it can serve others who may want to know.

Let R be an integral domain. An element $x \in R\backslash\{0\}$ is called primal if, for $y,z \in R\backslash\{0\}$, $x \mid_R yz$ implies that $x = rs$ where $r \mid_R y$ and $s \mid_R z$ (here $r \mid_R y$ stands for r divides y in R). An integrally closed integral domain with every (nonzero) element primal was called a Schreier domain, by the late Professor P.M. Cohn in [C] (Bezout rings and their subrings, Proc. Cambridge Philos. Soc. 64(1968), 251-264). I called R pre-Schreier if every nonzero element of R is primal [Z] (Zafrullah, On a property of pre-Schreier domains, Comm. Algebra 15(9)(1987), 1895-1920). While the name was new the notion had been discussed in [C] and in [MR] (McAdam and Rush, Schreier rings, Bull. London Math. Soc. 10(1978), 77-80). Call $x \in R\backslash\{0\}$ completely primal if each factor of x is a primal element. (My insistence on considering only nonzero elements is just to keep the theory simple to work with, otherwise one can follow the definitions in [C] or [Z].) In [C, Lemma 2.5] Cohn showed that in an integral domain the product of two completely primal elements is completely primal and in Theorem 2.6 he proved that if R is integrally closed, if S is a multiplicative set of R generated by completely primal elements and if R_S is Schreier then so is R and called it "Nagata's theorem for Schreier domains" In [Za] (Zafrullah, On Riesz groups, Manuscripta Math. 80(1993), 225-238) I redid this result, in more general terms, for Riesz groups and in Corollary 8 of [Za] I derived the result that you call my version of Cohn's Nagata type theorem. For clarity I restate this result here: Let R be an integral domain and let S be a set generated by completely primal elements of R. If R_S is a pre-Schreier domain then so is R. (On the same page I remark that this is not a new result, in that in the proof of [C, Theorem 2.6] Cohn does not use the property of R being integrally closed.)

Having thrown in necessary definitions, let me state the following result.

Proposition. Let R be a pre-Schreier domain with quotient field K and let $R[X]$ be the ring of polynomials over R. If R is not integrally closed then there is at least one nonzero element $r \in R$ such that r is not primal in $R[X]$.

Proof. Let us prove the contrapositive: If every nonzero element of R is primal in $R[X]$ then R is integrally closed. Let $\alpha \in K\backslash\{0\}$ be integral over R. Then α satisfies a polynomial $f(X) = X^n + a_{n-1}X^{n-1} + \ldots + a_0$ and so in $K[X]$ we can write $f(X) = (X-\alpha)(X^{n-1} + \ldots + \beta_0)$. Now there exist $s,t \in R\backslash\{0\}$ such that $s(X-\alpha) \in R[X]$ and $t(X^{n-1} + \ldots + \beta_0) \in R[X]$. Let $r = st$. Then

r is primal in $R[X]$ and $r \mid_{R[X]} [s(X-\alpha)][t(X^{n-1}+\ldots+\beta_0)]$ so by the definition of primal $r = ab$ where $a \mid_{R[X]} s(X-\alpha)$ and $b \mid_{R[X]} t(X^{n-1}+\ldots+\beta_0)$. This forces $a \mid_R s$, because $X-\alpha$ is a monic and $b \mid_R t$ because $(X^{n-1}+\ldots+\beta_0)$ is a monic. We claim that $\frac{s}{a}$ and $\frac{t}{b}$ are both units. This is because $f(X) = [\frac{s}{a}(X-\alpha)][\frac{t}{b}(X^{n-1}+\ldots+\beta_0)]$, which on comparing the coefficients, forces $(\frac{s}{a})(\frac{t}{b}) = 1$. Since $\frac{s}{a}$ and $\frac{t}{b}$ are both in R they must be units in R. But as $\frac{s}{a}(X-\alpha)$, $\frac{t}{b}(X^{n-1}+\ldots+\beta_0)$ belong to $R[X]$, we conclude that $(X-\alpha) \in R[X]$. This forces $\alpha \in R$. Thus every element integral over R is in R. (This of course means that R is integrally closed.)

So if R is a domain such that $R[X]$ is pre-Schreier then in particular every element of $R\backslash\{0\}$ is primal in $R[X]$ which makes R, pre-Schreier by the degree considerations and integrally closed by the above Proposition and hence a Schreier domain. Combining it with the result that if R is Schreier then so is $R[X]$ [C], we have the following result.

Corollary. Let R be a domain then $R[X]$ is pre-Schreier if and only if R is Schreier.

So, as I pointed out earlier, the flaw in your argument is that you are assuming that if R is pre-Schreier then the nonzero elements of R are primal in $R[X]$. Again for the record if R is a non-Schreier pre-Schreier domain then $R[X]$ is not pre-Schreier, nor Schreier for that matter.

Finally, if any of the readers wants to learn some more about pre-Schreier and Schreier domains here are a few more references for extra reading. This is a sort of off-hand collection. I have restricted it to papers where new results about (pre-) Schreier domains are proved or where primal elements are mentioned or used. If you come across a paper that deserves to be in this list, or if you have a comment that will improve this answer, do let me know at: mzafrullah@usa.net. I would be happy to revise the list and or the answer. (Muhammad)

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