QUESTION HD0902: How do you construct integrally closed domains that are not PVMD's? What's the simplest such example known?

ANSWER: I will first answer "How do you construct domains that are not PVMD's?" Then I will point out examples of certain integrally closed domains that fail to be PVMD's.

Using some known results we can construct non-PVMD domains. For example recall that a domain D is a PVMD if and only if D_M is a valuation domain for each maximal t-ideal M, see Griffin [G, Canad. J. Math. 19(1967) 710-722]. Thus a PVMD D is integrally closed; because (a) $D = \cap D_M$, where M ranges over maximal t-ideals of D, (b) a valuation domain is integrally closed and (c) intersection of integrally closed domains with the same field of fractions is integrally closed. So any integral domain that is not integrally closed is an example of a non-PVMD. Thus if V is a valuation domain of rank greater than 1 then the power series ring $V[[X]]$ is not a PVMD nor a GCD domain, because $V[[X]]$ is not integrally closed (see Proposition 13.11 of Gilmer's [Multiplicative Ideal Theory, Dekker, 1972]).

Now some examples of integrally closed integral domains that are not PVMD.

Recall that an integral domain D is a Schreier domain if D is integrally closed and D has the property (S): For $a, b, c \in D$, $a \mid bc$ implies that $a = \beta\gamma$ where $\beta \mid b$ and $\gamma \mid c$, in D. The Schreier domains were studied by Cohn in [C, Proc. Cambridge Philos. Soc. 64(1968), 251-264], where it was shown that a GCD domain is a Schreier domain and that if D is a Schreier domain then the polynomial ring $D[X]$ is also a Schreier domain. Also it is easy to see that (S) holds in direct limits of domains satisfying (S).

Now let $S \subseteq D\backslash\{0\}$ be a (saturated) multiplicative set in a domain D, let X be an indeterminate over D_S and consider $D^{(S)} = D + XD_S[X]$, the ring of polynomials from $D_S[X]$ with constant terms in D.The construction $D^{(S)}$ was studied by Costa, Mott and Zafrullah [CMZ, J. Algebra 53(1978), 423-439]. ([CMZ] can also ease a reader's familiarity with multiplicative Ideal theory.) Obviously $D^{(S)}$ is the direct limit (directed union) of polynomial extensions of the type $D[X/s]$ and thus if D is Schreier then so is $D^{(S)}$, as each of $D[X/s]$ is Schreier. In particular if D is a GCD domain then $D^{(S)}$ is Schreier.

It is known that a Schreier PVMD is a GCD domain (see e.g. Zafrullah [Z1, Comm. Algebra 15(1987), 1895-1920], following Theorem 3.6). So any Schreier domain that is not a GCD domain must not be a PVMD. This was used in [Z2, $D+XD_S[X]$ construction from GCD domains J. Pure Appl. Algebra 50(1988), 93-107] to construct a number of examples of non-PVMD domains. For now the following example will suffice.

Example. Let D be a discrete rank 2 valuation domain with maximal ideal pD and height one prime ideal Q and let S be generated by powers of p then $D + XD_S[X]$ is Schreier but not GCD because if q is a nonzero member of $Q = \cap p^n D$ then the only common factors of q and X are powers of p and every power of p divides both q and X. So GCD(q, X) does not exist and so $D + XD_S[X]$ is not a GCD domain.

Call a prime ideal P of a domain D essential if D_P is a valuation domain. A domain D is called essential if $D = \cap D_P$ where P varies over a set of essential primes of D. A PVMD is obviously essential. In [Z2] the $D+XD_S[X]$ construction was used, also, to give examples of essential domains that are not PVMD's, using the above argument. The earliest example of a non-PVMD essential domain was given by Heinzer and Ohm in [Canad. J. Math. 25 (1973), 856-861]. Fontana and Kabbaj [FK, Proc. Amer. Math. Soc. 132 (2004) 2529-2535] have also written on essential domains.

A domain D is called a v-domain if every nonzero finitely generated ideal A is v-invertible ($(AA^{-1})^{-1} = D$). A PVMD is a v-domain. For examples of v-domains that are not PVMDs see the paper by Fontana and Zafrullah [FZ, On v-domains: a survey]. This paper was written for a volume "Recent Developments in Commutative Algebra" to be published by Springer. You may consult the paper if and when it appears. For now you can find a pre-print at: www.lohar.com, in the section on Multiplicative Ideal Theory.

Anderson, Anderson and Zafrullah [AAZ, Arab. J. Sci. Eng. Sect. C Theme Issues 26 (2001), no. 1, 3–16] studied the question of when the $D + XD_S[X]$ construction is a PVMD. One of the characterizations is: $D^{(S)}$ is a PVMD if and only if D is a PVMD and (d, X) is t-invertible in $D^{(S)}$ for each $d \in D\backslash\{0\}$. A negation of this result or of some other equivalent conditions in Theorem 2.5 of [AAZ] can be used to construct non-PVMDs from PVMDs.

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