QUESTION HD0904: Let $A \subseteq B$ be an extension of integral domains, let X be an indeterminate over B and let $R = A + XB[X]$. Under what conditions is X (a) an irreducible element of R (b) a prime element of R?

ANSWER: Let us first note that $R = A + XB[X] = \{f(X) \in B[X] : f(0) \in A\}$ and that "$A \subseteq B$ is an extension of domains" requires that A is a subring of B. Let us also note that a nonzero nonunit element a of R is irreducible if and only if $a = bc$ implies that b is a unit or c is a unit. Thus a nonzero nonunit element $a \in R$ is reducible if and only if a is expressible as a product of two nonunits.

(a) Suppose that X is reducible in $A+XB[X]$. Then $X = f(X)g(X)$ where $f, g \in A + XB[X]$ and both are nonunits of $A + XB[X]$. Since $X, f(X), g(X)$ are all polynomials of $B[X]$ we conclude that, as $X = f(X)g(X)$, either the degree of $f(X)$ is zero or the degree of $g(X)$ is zero. Say the degree of $f(X)$ is zero then $f(X) = a \in A\backslash\{0\}$, where a is a nonunit of A, and the degree of $g(X)$ is one. So $g(X) = bX + c$. Setting $X = a(bX + c)$ and comparing the constants we have $c = 0$. Thus $X = a(bX)$ where a and bX are both nonunits of $A+XB[X]$. Comparing coefficients in $X = a(bX)$ we have $ab = 1$. Thus if X is reducible in R then there is a nonunit $a \in A$ such that $a^{-1} \in B$. Conversely if there is a nonunit $a \in A$ such that $a^{-1} \in B$ then $X = a(a^{-1}X)$ and so X is reducible. Thus we conclude that X is reducible in $A + XB[X]$ if and only if there is a nonunit a in A such that $a^{-1} \in B$. Equivalently X is irreducible in $A + XB[X]$ if and only if there is no nonzero nonunit in A with inverse in B.

Examples:

(1) X is irreducible in $A + XB[X]$ whenever A is a field, because A has no nonzero nonunits.

(2) X is irreducible in $A+XB[X]$ whenever B is a polynomial ring over A, because every unit of B is a unit of A.

(3) X is reducible in $Z + XL[X]$ where Z is the ring of integers and L is an integral domain containing p^{-1} for some prime p. In particular, X is reducible in $Z + XQ[X]$ (e.g. $X = 2(X/2)$).

It may be noted that given an extension $A \subseteq B$ of domains $U(B)\cap A = U(A)$ if and only if X is irreducible in $A+XB[X]$, where $U(D)$ denotes the set of units of the domain D. (The proof follows from the last line of the above answer.)

(b). Recall that a nonzero nonunit element p of a domain D is a prime if for $a, b \in D$, $p \mid ab$ implies that $p \mid a$ or $p \mid b$. It is well known that X is a prime in $A[X]$. So if $A = B$ then X is a prime in $A + XB[X]$. Conversely let $b \in B\backslash A$. Then $X \mid (bX)(bX)$ but as $b \notin A$, $X \nmid bX$ and so X fails to be a prime. Thus we conclude that X is a prime in $A + XB[X]$ if and only if $A = B$.

Examples:

(4) X is not a prime in $K + XL[X]$ if $K \subseteq L$ is an extension of fields and $K \neq L$.

(5) X is definitely not a prime in Examples (2) and (3).

Tiberiu Dumitrescu has provided a more sophisticated version of the above answer to part (b). For this recall that if M is an A-module then the A-idealization of M is the set $A\times M$ endowed with the usual component-wise addition and with multiplication defined by $(a_1, m_1)(a_2, m_2) = (a_1a_2, a_1m_2+a_2m_1)$.

You can read about idealization in Anderson and Winder's paper "Idealization of a module" [J. Commut. Algebra 1 (2009), no. 1, 3–56], where idealization of M by A is denoted by $A(+)M$. That, for an A-module M, $A(+)M$ is a ring is clear from the definitions of addition and multiplication. Note that for all $m \in M$, $(0,m)^2 = (0,0)$. So, we have the following statement.

(∗) $A(+)M$ is an integral domain if and only if A is an integral domain and $M = 0$. (If $A(+)M$ is an integral domain then for $m \in M$, $(0,m)^2 = (0,0)$ forces $(0,m) = (0,0)$ and so makes $M = (0)$. This makes $A(+)M \cong A$ and so A must be an integral domain. The converse is clear.)

Now $R/XR = \{(a + bX) \operatorname{mod}(X) : a \in A,\ b \in B\}$. The equality $(a + bX) \operatorname{mod}(X) = (a_1 + b_1X) \operatorname{mod}(X)$ forces $a - a_1 + (b - b_1)X \in (X)$ which forces $a = a_1$ and $b - b_1 \in A$. Thus R/XR can be identified with $A \times B/A$. Of course it is easy to check that B/A is an A-module. Next as $((a+bX) \operatorname{mod}(X))\,((a_1 + b_1X) \operatorname{mod}(X)) = (aa_1 + (ab_1 + a_1b)X) \operatorname{mod}(X)$ and as the addition $\operatorname{mod}(X)$ is given by $((a + bX) \operatorname{mod}(X)) + ((a_1 + b_1X) \operatorname{mod}(X)) = (a + a_1 + (b + b_1)X) \operatorname{mod}(X)$. It is now easy to see that R/XR is isomorphic to $A(+)B/A$ = the idealization of the A-module B/A. Now X is a prime in $R = A + XB[X]$ if and only if $R/XR \cong A(+)B/A$ is an integral domain, which by (∗) is possible if and only if $B/A = (0)$ which is possible if and only if $A = B$.

I am thankful to Tiberiu Dumitrescu for his help in finalizing this answer. He has also sent the following comments that I am sure will be of interest to those involved in more advanced considerations.

Let $A \subseteq B$ be an extension of domains and x a prime element of B with $xB \subseteq A$.

1. x is irreducible in A iff no nonunit of A becomes invertible in B.

Proof. If some nonunit a of A becomes invertible in B (with inverse b), then $x = a(bx)$ with a, bx nonunits of A.

Conversely, assume that $x = ac$ with a, c nonunits of A. As x is prime in B, we can suppose that $c = bx$ for some $b \in B$. So $x = abx$, hence $1 = ab$. Thus a is invertible in B.

2. x is prime in A iff $A = B$.

Proof. If b is in $B \backslash A$, then x divides $(bx)^2$ in A since $(bx)^2 = xb^2x$. Now if x were a prime in A then x dividing $(bx)^2 = (bx)(bx)$ should have implied that x divides bx. But as $b = bx/x$ is not in A, we conclude that x does not divide bx in A and so x is not a prime in A. The converse is obvious.

As some readers might have already noted that, specializing A to $A[X]$, B to $B[X]$ and x to X, we obtain the initial statements for $A + XB[X]$, as in (a) and (b).