QUESTION (HD1101) I have the following question. It is taken from the exercises in Kaplansky's book.

Let R be a Prufer domain. Let P be a finitely generated prime ideal. Prove that P is maximal.

Before the set of exercises, only 3 things have been proved. 1) Definition of Prufer domain, i.e every finitely generated ideal is invertible.

2) Invertible implies locally principal. 3) Localization of a prufer domain at a prime or maximal ideal is a valuation domain. Using these 3 facts, how can one give a proof of the above exercise.

ANSWER: Let me point out that the exercise should be: Let R be a Prufer domain. Let P be a finitely generated **nonzero** prime ideal. Prove that P is maximal. Let me also note that the definition of a Prufer domain is: every finitely generated **nonzero** ideal is invertible.

Here are some ways of dealing with the exercise.

(1) First let us fix the notation etc. R is an integral domain with K its quotient field. For any nonzero ideal A of R, $A^{-1} = \{x \in K, \, xA \subseteq R\}$. It is easy to see that $A^{-1} \supseteq R$.

Let R be Prufer and let P be a finitely generated nonzero prime ideal of R. To show that P is maximal we show that $(P, c) = R$ for every $c \in R \backslash P$. So let $c \in R \backslash P$.

Let $u \in (P, c)^{-1}$. Then uP, uc (and in particular) $ucP \subseteq D$. Since $uc \in R, ucP \subseteq P$ (P being an ideal). Next since $uP \subseteq R$, P is a prime ideal and since $c \notin P$ we conclude that $uP \subseteq P$.

Multiplying both sides of the last containment by P^{-1} we get $uPP^{-1} \subseteq PP^{-1}$ which implies $uR \subseteq R$ (since P is invertible, being finitely generated nonzero). But $uR \subseteq R$ forces $u \in R$. Since u was arbitrarily chosen this means that $(P, c)^{-1} \subseteq R$. But since already $(P, c)^{-1} \supseteq R$ we conclude that $(P, c)^{-1} = R$. Multiplying both sides of the last equation by (P, c) we get $(P, c)(P, c)^{-1} = (P, c)$. Since (P, c) is finitely generated and hence invertible we conclude that $(P, c) = (P, c)(P, c)^{-1} = R$.

(2) You need to know the following basic facts: Let R be a domain and let S be a multiplicative set in R. If $M \subseteq N$ are distinct prime ideals of R and if both M and N are disjoint with S then $MR_S \subseteq NR_S$ are distinct. Also if $P = pR$ is a principal prime ideal of R then p is an irreducible element of R that is if x is a nonunit and if x divides p then $xR = pR$. (These can be gleaned from earlier sections of the book.)

Now let R be Prufer and let P be a finitely generated nonzero prime ideal of R and suppose by way of contradiction that P is not maximal. Let M be a maximal ideal (properly) containing P. By your 3), R_M is a valuation domain and by your 2) PR_M is a principal ideal, say $PR_M = pR_M$. Now $pR_M \subseteq MR_M$. Claim: $PR_M = MR_M$. For if $x \in MR_M$ then, because R_M is a valation domain $p \mid x$ or $x \mid p$ in R_M. If $p \mid x$ then $x \in pR_M = PR_M$ and if $x \mid p$ then since x is a nonunit and p is a prime $xR_M = pR_M$ forcing $x \in pR_M = PR_M$. Thus $PR_M = MR_M$, but this contradicts the assumption that P is properly contained in M.