QUESTION (HD 1103): Let R be a commutative ring. Can we say anything nice about R if we know that the set of zero divisors of R is a prime ideal? (This interesting question was asked by Viji Thomas from TIFR(Tata Institute of Fundamental Research), Mumbai, India)

ANSWER: Let me first make clear what I mean by a zero divisor and in what context.

Let R be a commutative ring with $1 \neq 0$. An element $x \in R$ is a zero divisor if there is a nonzero $y \in R$ such that $xy = 0$. Thus 0 is a zero divisor and R is an integral domain if R has no nonzero zero divisors. An element $x \in R$ is a nonzero-divisor if there is no nonzero $y \in R$ such that $xy = 0$. A nonzero-divisor is also called regular. It is easy to establish that the set $\rho(R)$ of regular elements of a ring R is multiplicative and saturated. So, $R \backslash \rho(R)$ is a union of prime ideals. Indeed $R \backslash \rho(R) = Z(R)$ the set of zero divisors of R. So $Z(R)$ is a union of prime ideals of R.

Lemma A. Let $\{P_\alpha\}_{\alpha \in I}$ be a family of prime ideals of a commutative ring R and let $K = \cup P_\alpha$. Then K is a prime ideal if and only if K is an ideal. Consequently $Z(R)$ is a prime ideal if and only if $Z(R)$ is an ideal.

Note that $xy \in K$ implies that $xy \in P_\alpha$ for some α. But P_α being a prime $x \in P_\alpha \subseteq \cup P_\alpha$ or $y \in P_\alpha \subseteq \cup P_\alpha$. Thus K is a prime ideal as soon as it is an ideal. The consequently part is evident.

Note A$'$: It may be noted that as K is a union of ideals the only thing that keeps K from becoming an ideal is the property of being closed under addition. Thus Lemma A can be restated as: Let $\{P_\alpha\}_{\alpha \in I}$ be a family of prime ideals of a commutative ring R and let $K = \cup P_\alpha$. Then K is a prime ideal if and only if K is closed under addition. Consequently $Z(R)$ is a prime ideal if and only if $Z(R)$ is closed under addition.)

Call rings R with $Z(R)$ an ideal the ζ (zeta) rings and note that an integral domain is a zeta ring.

Proposition B. A ring R is a zeta ring if and only if the sum of every pair of zero divisors is a zero divisor.

Proof. If R is a zeta ring and so $Z(R)$ is an ideal and so the sum of every pair of zero divisors is a zero divisor. Conversely we show that if for every pair of zero divisors x, y the sum $x + y$ is a zero divisor then $Z(R)$ is an ideal and hence a prime ideal. For this note that if $x \in Z(R)$ then for each $r \in R$ we have $rx \in Z(R)$. So, in particular, $-x \in Z(R)$. So for all $x, y \in Z(R)$, $x - y \in Z(R)$. So, $Z(R)$ is a subgroup of R under addition and it is straightforward to see that $Z(R)$ is an ideal.

Note B$'$: I could have used Note A' in the proof of the conversely part of Proposition B, but nothing beats a direct proof.

Proposition C. R is a zeta ring if and only if for all $r \in \rho(R)$ and for all $z \in Z(R)$, $r + z \in \rho(R)$.

Proof. Suppose R is a zeta ring and for some $r \in \rho(R)$ and $z \in Z(R)$, $r + z$ is a zero divisor, i.e. $r + z \in Z(R)$. Then as $Z(R)$ is an ideal we have $r \in Z(R)$ a contradiction. Conversely, suppose that for all $r \in \rho(R)$ and for all $z \in Z(R)$, $r + z \in \rho(R)$ and suppose that $Z(R)$ is not closed under addition, i.e., say there exist $u, v \in Z(R)$ such that $u + v \notin Z(R)$. Then $u + v = r \in \rho(R)$. But then $u = r - v \in \rho(R)$ contradicting the assumption that $u \in Z(R)$.

Proposition D. If R is a zeta ring then $T(R)$ the total quotient ring of R is a quasi-local ring.

Proof. Note that $T(R) = \{\frac{x}{r} : x \in R \text{ and } r \in \rho(R)\}$ and as $Z(R)$ is a prime ideal $R_{Z(R)} = \{\frac{x}{r} : x \in R, r \notin Z(R)\} = \{\frac{x}{r} : x \in R, r \in R \backslash Z(R)\} = \{\frac{x}{r} : x \in R, r \in \rho(R)\}$. Comparing, we get $T(R) = R_{Z(R)}$ a quasi-local ring.

It is easy to see that if R is a zeta ring then $Z(T(R)) = Z(R_{Z(R)}) = Z(R)R_{Z(R)}$.

Proposition E. Let R be a quasi-local ring with maximal ideal M. Then $T(R) = R$ if and only if $Z(R) = M$.

Note that $Z(R) = M$ if and only if every regular element of R is a unit if and only if $T(R) = R$.

These are just a few remarks. If my health holds, I might have something more to say on this topic.

Notes: (1) Dan Anderson has noted that in a zeta ring R, $Z(R)$ does not have to be the nilradical i.e. intersection of all prime ideals of R. His example: Let R be a two dimensional regular local ring with maximal ideal M and let x be a prime element in R. Then R/xM is a local zeta ring. Obviously $Z(R/xM) = M/xM$. Now take y a prime element in R such that y is not an associate of x. Then, $y + xM$ is a zero divisor, as $(y + xM)(\ x + xM) = xM$, but $y + xM$ is not nilpotent. (Recall that the every nonzero element of the nilradical must be nilpotent.)

(2) George Bergman observed that: If R is an integral domain , and P any nonzero prime ideal of R, then R can be embedded in a zeta-ring $R^{'}$ such that $R \cap Z(R^{'}) = P$. Namely, let $R^{'} = R[x]/(Px + x^2R[x])$. The zero-divisors of $R^{'}$ are simply all elements whose "constant terms" are members of P. George also pointed out that if R is a ring with a prime ideal P such that $Z(P) \subseteq P$ even then $R^{'} = R[x]/(Px + x^2R[x])$ will have the property that $R \cap Z(R\prime) = P$.

(3) I am thankful to David Dobbs who thinks that Proposition C is perhaps a new result. As far as I am concerned, I think all the "results" included in this answer are so elementary that it is hard to imagine that no one has got any one of them before.

(4) I am indebted to Tiberiu Dumitrescu, who read the answer critically and pointed out several errors that I am prone to make. Tiberiu also had the following to say.

a. In a Noetherian ring R, $Z(R)$ is the (finite) union of the ideals in $Ass(R)$. A finite union of ideals is an ideal iff this union is one of the ideals.

So, a Noetherian ring R is zeta iff $Ass(R)$ has a biggest ideal Q. In particular, $Q = Ann(q)$ for some q. Summing up, a Noetherian ring is zeta iff the set of annihilator ideals has a biggest element (that ideal is automatically prime). In this connection looking at the example in Note (1) we see that $Z(R/xM) = M/xM = ann(x + xM)$.

b. A trivial remark: a zero-dimensional ring is zeta iff it is quasi-local, because in this case the ring equals its total quotient ring.

I am grateful to each of the advisors/contributors.

The above material was posted around November 16, 2011.

(5) Let's look into the question: If R is a zeta ring must $R[X]$ be? (Here X is an indeterminate over R.)

Recall that McCoy's Theorem states that a polynomial $f(x) \in R[x]$ is a zero divisor if and only if there is a nonzero element $r \in R$ such that $rf(x) = 0$. But the trouble with this "if and only if" statement is that it does not tell us what happens in case f is a polynomial over a zeta ring R with all its coefficients from $Z(R)$.It is patent that if f is such that all the coefficients of f are zero divisors then f may not be a zero divisor. Though the classical example is somewhat unrelated, it gives the idea: Over the ring $Z \times Z$, where addition and multiplication are defined componentwise, the elements $(0, 1)$ and $(1, 0)$ are zero divisors, the polynomial $(0, 1) + (1, 0)X$ is not a zero divisor, as Tiberiu pointed out. Of course, obviously $Z \times Z$ is not a zeta ring, as $(0, 1) + (1, 0) = (1, 1)$ the identity of $Z \times Z$. But still being on the side of caution we look into the situation where all the polynomials are zero divisors; we consider the case when $R[X]$ is a zeta ring. Indeed if $R[X]$ is a zeta ring then $f = \sum f_i X^i \in Z(R[X])$ implies that there is a nonzero $r \in R$ such that $rf_i = 0$. This seems to dictate the following condition (S) on R : For every finite set of elements $a_1, a_2, ...a_n \in Z(R)$ there is a nonzero $r \in R$ such that $ra_i = 0$. Call a ring R satisfying (S) a special ring. Obviously a special ring R is a zeta ring. Now $Z(R[X])$ being a prime ideal implies that $Z(R[X]) \cap R$ is a prime ideal and it is easy to establish that $Z(R[X]) \cap R = Z(R)$.So, the following proposition can be proved.

Proposition F. Let X be an indeterminate over a commutative ring R. Then R is a special ring if and only if $R[X]$ is, and that case $Z(R[X]) = Z(R)[X]$.

It would be interesting and instructive to find out if all zeta rings are special or not.

Next suppose that R is a ring, I a proper ideal of R and suppose that R/I is a zeta ring. Then $Z(R/I)$ is an ideal and so is closed under addition. That is if $\overline{x_i} = x_i + I \in Z(R/I)$, for $i = 1, 2$, then $\overline{x_1} + \overline{x_2} = (x_1 + I) + (x_2 + I) = x_1 + x_2 + I \in Z(R/I)$. Now $x_i + I \in Z(R/I)$ means that there are $t_i \in R\backslash I$ such that $(x_i + I)(t_i + I) = I$. Further translating $x_i + I \in Z(R/I)$, means that there are $t_i \in R\backslash I$ such that $x_i t_i \in I$. Similarly $x_1 + x_2 + I \in Z(R/I)$ means that there exists a $t \in R\backslash I$ such that $(x_1 + x_2)t \in I$. Thus given that R/I is a zeta ring and a pair $x_i \in R$ if there are $t_i \in R\backslash I$ such that $x_i t_i \in I$ then there is a $t \in R\backslash I$ such that $(x_1 + x_2)t \in I$. The process involves translations and so can be reversed. Hence we have the following proposition.

Proposition G. Let R be a commutative ring and I a proper ideal of R. Then R/I is a zeta ring if and only if the following condition holds:

For all $x, y \in R$ if there are $r, s \in R\backslash I$ such that $xr, ys \in I$ then there is $t \in R\backslash I$ such that $(x + y)t \in I$.

(6) While the following result is well known we shall verify Proposition G, using the well known proof of it.

Proposition H. Let R be a commutative ring and I a primary ideal. Then R/I is a zeta ring.

Proof. Because $\sqrt{I} = P$ is a prime the nilradical of R/I is P/I. Now let $x \in R$ be such that $x + I \in Z(R/I)$, so there is $r \in R\backslash I$ such that $(x + I)(r + I) = I$. So $xr \in I$. Now as r is not in I, $x^n \in I$ for some n . Thus for all x such that $x + I \in Z(R/I)$, $x^n \in I$ for some n. Thus $Z(R/I) \subseteq P/I$ the nilradical of R/I. This forces $Z(R/I) = P/I$ which is an ideal and so R/I is a zeta ring.

Verification: Note that for every $x \in R$ such that $x + I \in Z(R/I)$ we have, for a least n, $x^n \in I$. So $x^{n-1} = r$ (of Proposition G). If $n = 1$, we get r a unit. Next the same happens for a y such that $y + I \in Z(R/I)$. So, say, there is $y^{m-1} = s \in R\backslash I$ and the proof of Proposition H gives a k such that $(x+y)^k \in I$ such that $t = (x + y)^{k-1}$.

Corollary (to Proposition G) H$'$. If I is a nonzero ideal of a valuation domain V, then V/I is a zeta ring.

Proof. Let $a, b \in V$ be such that $a + I$, $b + I \in Z(V/I)$. Then there exist $r, s \in V\backslash I$ such that $ar, bs \in I$. Now note that as V is a valuation domain $r \mid s$ or $s \mid r$, say $r \mid s$. Then $as, bs \in I$ and hence, as I is an ideal, $as+bs = (a+b)s \in I$.

The statement and a proof for this corollary was provided by A. Mimouni. Abdeslam also told me that it can be found as Lemma 3.1.9 on page 40, in Fontana, Huckaba and Papick's book [Papick, Prufer Domains, Monographs and Textbooks in Pure and Applied Mathematics, 203. Marcel Dekker, Inc., New York, 1997.].

Another proof follows from a more general result:

Proposition H" : Let R be a quasi-local ring such that $Spec(R)$ is totally ordered and if I is a nonzero ideal of R. Then R/I is a zeta ring.

The proof follows from the fact that $Spec(R/I)$ is totally ordered and $Z(R/I)$ is a union of prime ideals.

Lemma 3.1.9 of that book reads as: Let I be a nonzero ideal of a valuation domain V, then $I : I = V_P$ where P is the prime ideal of all zero divisors on V/I.

It may be noted that if V is a valuation domain and I a nonzero ideal of V then $Z(V/I)$ may be different from $N(V/I)$ the nilradical of V/I. This was indicated in Example 3.16(b) of Anderson, Badawi and Dobbs' paper [Bollettino U.M.I. (8) 3B(2000), 535-545.] To see this let x be a nonzero element of a suitable valuation domain V with maximal ideal M such that the minimal prime ideal Q of xV is property contained in M. Let $r \in M\backslash xV$ then $r \mid x$ and $\frac{x}{r} \notin xV$. Now $(r + xV)(\frac{x}{r} + xV) = xV$. This way we see that $Z(V/xV) = M/xV$, On the other hand $N(V/xV) = Q/xV$.

(7) Next arises the following question

Question: If I is an ideal such that $\sqrt{I}$ is a prime, must $Z(R/I)$ be a zeta ring, when I is not a primary ideal?

There do exist non-primary ideals with radical a prime. Some of the examples can be derived from the following result. But first recall from Cohn [Math. Proc. Cambridge Philos. Soc. 64 (1968), 251-264.] that a nonzero x in a domain is called primal if $x \mid ab$ implies $x = rs$ where $r \mid a$ and $s \mid b$.

Proposition K. Let P be a prime ideal of an integral domain D and x a primal element of D such that every nonunit factor of x is in P. Then D/xP is a zeta ring.

Proof. We show that $Z(D/xP) = P/xP$. For this note that $Z(D/xP) \supseteq P/xP$ obviously. For the reverse containment let $a \in D$ such that $a + xP \in Z(D/xP)$ and suppose, by way of getting a contradiction that $a \notin P$. Then there is $r \in D\backslash xP$ such that $(a + xP)(r + xP) = xP$, that is $ar \in xP \subseteq (x)$. Now $x \mid ar$ and x is primal. So $x = uv$ where $u \mid a$ and $v \mid r$. Since $a \notin P$, u must be a unit; forcing $x \mid r$. From this and the fact that $ar \in xP$ we conclude that $a\frac{r}{x} \in P$. Since $a \notin P$ we have $\frac{r}{x} \in P$. This gives $r \in xP$ a contradiction.

Corollary L. Let P be a prime ideal of a domain D and p a prime element in P. Then for every natural number n, D/p^nP is a zeta ring with $Z(D/p^nP) = P/p^nP$.

Corollary M. In $D = K[X, Y]$, where K is a field. For the ideal $I = (X^2, XY)$ (which is known to not to be primary yet having radical the prime (X)), D/I is a zeta ring with $Z(D/I) = (X, Y)/(X^2, XY)$.

The proof follows from the fact that $(X^2, XY) = X(X, Y)$. Note that in this case $Z(D/I) \supsetneq N(D/I)$.

Tiberiu Dumitrescu has offered the following improvement on Proposition K.

Proposition N. Let D be a domain $I \subseteq P$ ideals with P prime and I invertible ideal such that $(I, x)_v = D$ for every x outside P. Then the set of zero-divisors in D/IP is P/IP.

Proof. Let $x, y \in D$ such that $xy \in IP$ and $x \notin P$ and $y \notin IP$. As $xy \in I$, and $x \notin P$, we have $(I, x)_v = D$, so $y \in I$. Then $x(yI^{-1}) \subseteq P$ and $yI^{-1} \subseteq D$ forces $yI^{-1} \subseteq P$, so $y \in IP$, a contradiction.

(8) On December 2, 2011, Tiberiu Dumitrescu informed me that Laszlo Fuchs had studied zeta rings. This needs some introduction. In his paper [On primal ideals, Proc. Amer. Math. Soc. 1 (1950), 1-6]. The dazzling Laszlo describes a primal ideal as an ideal I in a way that can be translated into: An ideal I of a ring R is a primal ideal if for all $x, y \in R$ whenever there are $r, s \in R\backslash I$ such that $xr, ys \in I$, there is $t \in R\backslash I$ such that $(x + y)t \in I$ (cf Proposition G). He (Fuchs) proved that, in our terminology, an ideal I of a ring R is primal if and only if R/I is a zeta ring. So, in (6) and (7), whatever results are proved for zeta rings are also results about primal ideals. Next let us see what zeta rings really are. Note that, by Proposition G above, R/I is a zeta ring if and only if and only if I is a primal ideal. Now for any ring R, we have $R \simeq R/(0)$. Thus R is a zeta ring if and only if (0) is a primal ideal of R. So zeta rings were essentially studied by Laszlo Fuchs.

(9) On December 4, 2011, Tiberiu sent me two papers. Of these one is by Ahmed Yousefian Darani [An. Ştiinţ. Univ. "Ovidius" Constanţa Ser. Mat. 18(2), 2010, 59-72]

In this paper Darani discusses something like the property (S), discussed in (5). That is he calls an R-module M super-coprimal if for each finite set $\{a_1, ..., a_n\} \subseteq Z_R(M)$ there is a nonzero $r \in R$ such that $ra_i = 0$. While reading this paper I found Toma Albu and Patrick Smith's [Rev. Roumaine Math. Pures Appl., 54 (4)(2009), 275-286]. In this paper a module M is called primal if $Z_R(M)$ is an ideal of R and they call a submodule N of a module M primal if the quotient module M/N is coprimal. From this one "can" derive results about zeta rings as coprimal rings.

The other paper that Dumitrescu sent me is by Joong Ho Kim [Bull. Korean Math. Soc. 30(1993), No. 1, pp. 71-77]. To see what this paper says let's recall that a ring R has property $(*)$ if every finitely generated ideal of R consisting entirely of zero divisors of R has a nonzero annihilator, Huckaba [Commutative rings with zero divisors, Dekker, 1988]. The author shows: Let X be an indeterminate over R and A a primal ideal of R. Then R/A satisfies $(*)$ if and only if $A[X]$ is a primal ideal of R. Note that for a zeta ring the property $(*)$ is the same as the property (S) derived in (5) above; only (S) forces the ring to be a zeta ring.

(10) On December 5, 2011. Bruce Olberding sent me, at my request, a list of papers related to the study of primal ideals. Here are the references:

(i) Fuchs, Heinzer and Olberding [Trans. Amer. Math. Soc. 357 (2005), no. 7, 2771–2798].

(ii) Fuch, Heinzer and Olberding [Trans. Amer. Math. Soc. 358 (2006), no. 7, 3113–3131].

(iii) Fuchs and Mosteig, [J. Algebra 252 (2002), no. 2, 411–430]

(11) Looking at all the material, it seems to me that everyone has been studying primal ideals and/or their module counterparts skirting the rings that I got to call the zeta rings. I am not complaining, just commenting that this is how things work. A researcher would pay more attention to the concepts that are of interest to him/her. So Viji Thomas' question was legit and I dare say spot on.

Finally I thank, again, all the contributors and advisors.

Muhammad Zafrullah