QUESTION: (HD 1104) In HD1103 you have used two terms: primal ideal and primal element. Are they related? Can I say that the principal ideal generated by a primal element is a primal ideal? (I recall that some authors call an element x of a domain R primary if xR is a primary ideal.)

ANSWER: I am afraid the answer is generally **no**, since the definition of a primal element is different from the definition of a primal ideal. Of course, with reference to HD1103, an ideal I of a ring R is primal if and only if R/A is a zeta ring. On the other hand, as indicated in HD1103, an element x of an integral domain D is primal if for all $a, b \in D$, $x \mid ab$ implies that $x = rs$ where $r \mid a$ and $s \mid b$. This notion was introduced by P.M. Cohn in his paper "On Bezout rings and their subrings" [Math. Proc. Cambridge Philos. Soc. 64 (1968), 251-264]. But this was not all that Cohn did in that paper.

Paul Cohn defined a Schreier ring as an integral domain R such that R is integrally closed and every (nonzero) element of R is primal. Among many things he showed that a GCD domain is Schreier. So, in particular a unique factorization domain is a Schreier domain. Now let K be a field and X, Y be two indeterminates over K. It is well known that $R = K[X, Y]$ is a UFD. So, $XY \in R$ is a primal element. Consider R/XYR and note that $X, Y \in Z(R/XYR)$. To see that $X + Y \notin Z(R/XYR)$ let there be $a \in R$ such that $(X + Y)a \in XYR$. So there is $t \in R$ such that $Xa + Ya = XYt....(A)$.

Rewriting (A) as: $Xa = XYt - Ya = (Xt - a)Y$ and noting that Y and X are coprime we conclude that $Y \mid a$. Similarly rewriting (A) as $Ya = XYt - Xa = (Yt - a)X$ we conclude that $X \mid a$. Now again X, Y being coprime and dividing a implies that XY divides a. This of course shows that $(X + Y)a = 0 \bmod XYR$ implies that $a = 0 \bmod XYR$ establishing that $Z(R/XYR)$ is not closed under addition, R/XYR is not a zeta ring, and hence XYR is not a primal ideal of R. So a primal element does not generate a primal ideal.

On the other hand, as we know from Anderson and Mahaney's [J. Pure Appl. Algebra 54(1988) 141-154], an element x in a ring R is primary if xR is a primary ideal. As we have seen in Proposition H of HD1103, if I is a primary ideal of R then R/I is a zeta ring. Next a primary element may be a primal element if R is a Schreier domain. So it's a sort of a mixed bag situation and you have to stick with the definition; especially if you are mixing the two notions to get a new result. We showed in Proposition K of HD1103 that if P is a prime ideal of a domain R and x a primal element of R such that every nonunit factor of x is in P then xP is a primal ideal.

The last result, in the above paragraph, can be strengthened a little, but we need some terminology for that. If I is a primal ideal of a ring R then the prime ideal P of R such that $P/I = Z(R/I)$ was called an adjoint ideal, by Fuchs in [Proc. Amer. Math. Soc. 1 (1950), 1-6]. With this terminology we state the following result.

Proposition. Let I be a primal ideal of a domain R with P the adjoint prime ideal of I. If x is a primal element of R such that every nonunit factor of x is in P, then xI is a primal ideal with adjoint P.

Proof. Note that $Z(R/xI) \supseteq P/xI$. This is because for each $a \in P$ we have

$r \in R\backslash I$ such that $ar \in I$. But then $arx \in xI$ where $rx \notin xI$. Thus for each $a \in P$, $a + xI \in Z(R/xI)$. For the reverse containment, let $a + xI \in Z(R/xI)$ and suppose, by way of contradiction, that $a \notin P$ then there is $r \in R\backslash xI$ such that $ra \in xI$. This gives $x \mid ra$ and so $x = uv$ where $u \mid r$, $v \mid a$ and as a is assumed to be not in P v is a unit and so $x \mid r$. Thus $a\frac{r}{x} \in I \subseteq P$. Because $a \notin P$, a is a nonzero divisor $\bmod I$ whence $a\frac{r}{x} \in I$ implies that $\frac{r}{x} \in I$ which gives $r \in xI$ a contradiction.

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