QUESTION: (HD 1105) Must an almost factorial domain be locally factorial?

ANSWER: The answer is no. Yet, instead of pointing to an example right away, let me make some introductory remarks to make the example accessible to you and the general readers. In the common terminology a Krull domain D is called an almost factorial domain if the divisor class group of D is torsion. As indicated in Fossum's book [F] the notion of almost factoriality was introduced by U. Storch in [S]. According to Proposition 6.8 of [F] a Krull domain D is almost factorial if and only if for each pair of elements $a, b \in D$ there is a natural number n such that $a^n D \cap b^n D$ is principal if and only if for every height one prime ideal P of D, there is a natural number n such that $P^{(n)}$ is principal. Here $P^{(n)} = P^n D_P \cap D$ represents the nth symbolic power of P. It is easy to establish that for every ideal P in a Krull domain D we have $P^{(n)} = (P^n)_v$.

A bit of introduction to the operation v is in order here. You can skip it if you are familiar with the jargon. Let $F(D)$ denote the set of nonzero fractional ideals of an integral domain D with quotient field K. For each $A \in F(D)$ define $A^{-1} = \{x \in K : xA \subseteq D\}$, obviously $A^{-1} \in F(D)$. Define $v : F(D) \to F(D)$ by $A \mapsto A_v = (A^{-1})^{-1}$. This is an example of a star operation called the v-operation. We shall freely use the properties of the v-operation. A reader in need of a review may look up sections 32 and 34 of Gilmer [G]. We call $A \in F(D)$ a v-ideal if $A = A_v$. A v-ideal is also called a divisorial ideal. From section 3 of [F], we conclude that D is a Krull domain if and only if D is completely integrally closed and satisfies ACC on divisorial ideals. An ideal I of a Krull domain is a maximal divisorial if and only if I is a height one prime. If $X^{(1)}(D)$ denotes the set of height one primes of a Krull domain D then for all $P \in X^{(1)}(D)$ D_P is a valuation domain and $D = \bigcap_{P \in X^{(1)}(D)} D_P$ is a locally finite intersection. Of course if D is Krull, $P \in X^{(1)}(D)$ and $x \in D \backslash P$ then $(x, P)_v = D$ because P is a maximal divisorial ideal. Finally, every invertible ideal is divisorial as $A = (A^{-1})^{-1}$.

Next an integral domain D is said to be locally factorial if D_M is a UFD for every maximal ideal M of D. Generally, locally factorial domains may not have very nice properties, but a locally factorial Krull domain has some very interesting properties, as shown by Anderson in [A]. According to Theorem 3.1 of [A] a Krull domain D is locally factorial if and only if every height one prime ideal of D is invertible.

Of course if I can show you an example of a Krull domain D such that $Cl(D)$ is finite such that a height one prime ideal P is not invertible that would answer the question, in light of Theorem 3.1 of [A]. It so happens that there is such an example available in [F] at page 51. (My treatment of it is slightly different from [F].)

Example A. The ring $A_F = \mathcal{C}[X_1, X_2, X_3]/(X_1^2 + X_2^2 + X_3^2)$ is a Krull domain with $Cl(A_F) = Z/2Z$. (Here $\mathcal{C}$ denotes the field of complex numbers and Z the ring of integers.) A_F contains a non-invertible prime divisorial ideal.

Note that A_F is an integral domain because of the irreducibility of $F = X_1^2 + X_2^2 + X_3^2$. Also A_F is Noetherian, obviously. Next A_F is integrally closed by Proposition 11.2 of [F]. That an integrally closed Noetherian domain is Krull is well known.

Setting $X = X_1+iX_2$ and $V = X_1-iX_2$. This makes $A_F = \mathcal{C}[X,V,X_3]/(XV+X_3^2)$, and so $A_F = \mathcal{C}[x,v,x_3]$ where $xv + x_3^2 = 0$. Next $A_F[v^{-1}] = \mathcal{C}[x_3,v,v^{-1}]$. Since x_3, v are algebraically independent over $\mathcal{C}$, $\mathcal{C}[x_3,v]$ and hence $\mathcal{C}[x_3,v,v^{-1}]$ is factorial so $Cl(A_F[v^{-1}] = 0$. Now by Nagata's Theorem, especially Corollary 7.2 of [F], $Cl(A_F) \to Cl(A_F[v^{-1}])$ is a surjection and so every non-principal height one prime of A_F is in the class of the height one prime ideal that contains v. To see that there is only one such prime ideal we consider $A_F/(v) = \mathcal{C}[x,v,x_3]/(v)$ and note that the only nonzero zero divisors of this ring are the multiples of x_3 (because $x_3^2 = -xv$) and so $Z(\mathcal{C}[x,v,x_3]/(v)) = (x_3,v)/(v)$ which is an ideal and hence a prime ideal (cf zeta rings in HD1103 http://www.lohar.com/mithelpdesk/hd1103.pdf). This forces (x_3,v) to be a prime ideal. Being a minimal prime ideal of (v) in the Noetherian Krull domain (x_3,v) is of height one and divisorial. (Another way of showing that (x_3,v) is divisorial of height one is to note that $(x_3,v) = (x_3) :_{A_F} (x)$.)

Next let $P = (x_3,v)$. Then $P^2 = (x_3^2, x_3v, v^2) = (xv, x_3v, v^2)$ (because $x_3^2 + xv = 0$). So $P^2 = v(x,x_3,v) \subsetneq (v)$, because (x,x_3,v) does not contain a unit. Also as $x \notin (x_3,v)$ which is a maximal divisorial ideal, being of height one, we conclude that $(x,x_3,v)_v = A_F$. Thus we have $P^{(2)} = (P^2)_v = (v(x,x_3,v))_v = v(x,x_3,v)_v = (v)$. We shall use this to prove that $Cl(A_F) = Z/2Z$. Let us first show that P is not invertible. Obviously if P is invertible then so is P^2 and so is divisorial. But then $P^2 = (P^2)_v = (v)$ and this contradicts the already observed fact that $P^2 \subsetneq (v)$. Now as P is not invertible P cannot be principal either. So $Cl(A_F) \neq 0$. This leaves us with $Cl(A_F) = \{[P];\ [P]^2 = 0\}$ forcing $Cl(A_F) = Z/2Z$.

Note1. An integral domain D is called an almost GCD (AGCD) domain if for each pair $a,b \in D$ there is a natural number $n = n(a,b)$ such that $a^nD \cap b^nD$ is principal. The AGCD domains were introduced in [Z] and further studied in [AZ]. It was shown in [Z] that an integrally closed AGCD domain is a PVMD with torsion t-class group. The point of mentioning AGCD domains is that a Krull domain is a PVMD and so a Krull domain is almost factorial if and only if it is an AGCD domain. So example A above also shows that an integrally closed AGCD domain may not be locally a GCD domain.

Note 2. Example A also answers the following question: Is there a divisorial ideal A whose square is not divisorial? The prime P in Example A is such a divisorial ideal. (Someone asked this question recently.)

I wish everyone all the happiness during the next year.

Muhammad Zafrullah,

December 31, 2011

Note 3. Dan Anderson wants me to warn the readers that some authors call an integral domain D "locally factorial" if for all $d \in D\backslash\{0\}$, $D[1/d]$ is a factorial domain and that that kind of locally factorial is not being discussed here. You can look up [AA] to have an idea of the locally factorial domains that

Dan is talking about.

Note 4. David Dobbs made the following comment which may be of interest to aspiring homological algebraists:

Here's a bit of philosophy as to why the square of a divisorial should not be expected to be divisorial very generally, from my homologically driven intuition. Reflexives are the module-theoretic version of v-ideals. Studying the inverses or dual modules relevant to these is easily done by direct sum arguments when the players are finitely generated projectives (i.e., invertible if nonzero ideals of a domain), but there is no Aristotelian sufficient cause to expect good behavior when a divisorial is far from invertible. When P is far from projective, P may also be far from flat, and so P^2 may be quite different from the tensor product (taken over the ambient domain) of P with itself. So, while that tensor product is finitely generated projective when P is, there's no reason to expect that the behavior of the tensor product can predict the behavior of its homomorphic image, P^2, when P is far from the homological paradigms.

All that the above goes to show is that I would have bet on the likelihood of the existence of the kind of example that you gave, which is not at all the same as actually providing the example, of course.

Thank you David. Now I wish I had produced an original example. Though I think there should be plenty of examples of ideals I such that I is divisorial and I^2 is not.

January 5, 2012:

Tiberiu Dumitrescu has come up with two examples of prime divisorial ideals P with non-divisorial squares.

(1) Take the domain $D = Z[tX, X^2]$ where $t = \sqrt{2}$). D is a Krull domain because $D = Q[tX] \cap Z[t][X]$. D is two-dimensional because $D \subseteq Z[t][X]$ is an integral extension. Set $P = (tX, X^2)$. As $D/P = Z$ and D is two-dimensional, it follows that P has height one, so P is a v-ideal. It can be proved directly that $P^{-1} = X^{-2}P$, so $P = P_v$. Now $P^2{=}X^2(2, tX, X^2) \subsetneq (X^2)$. As 2 is not in P, we get $(2, tX, X^2)\_v = D$, so $(P^2)_v = (X^2)$.

(2) Take the domain $D = Z[X, 4/X]$. D is a 2-dimensional Krull domain (cf. [AA1]). Set $P = (2, X)$. As $D/P = (Z/2)[X^{-1}]$ and D is two-dimensional, it follows that P has height one, so P is a v-ideal.

Now $P^2 = X(4/X, 2, X) \subsetneq (X)$. As $4/X$ is not in P, we get $(4/X, 2, X)_v = D$, so $(P^2)_v = (X)$.

I recommend that an interested reader may read first few pages (at least) of [AA1]. The paper is a very good source of examples.

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