QUESTION: (HD 1201) How do primary, quasi primary and primal ideals relate? In particular if I is an ideal such that $\sqrt{I}$ is a prime must I be a primal ideal?

(I asked part of this question in HD1103, and Bruce Olberding provided such an elaborate answer that it necessitated the earlier part of the question.)

ANSWER: Let's start with relevant definitions first. The rings here are commutative with $1 \neq 0$.

An ideal I of a ring R is primary if, for $a, b \in R$, $ab \in I \Rightarrow a \in I$ or $b^n \in I$. If I is primary $\sqrt{I}$ is a prime ideal but $\sqrt{I}$ being prime does not mean that I is a primary ideal. An easy example is the following.

Example A. The ideal $I = (X^n, X^{n-1}Y)$, where $n > 1$, in the polynomial ring $K[X, Y]$ over a field K is not primary but $\sqrt{I}$ is a prime ideal. Here $X^{n-1}Y \in I$, $X^{n-1} \notin I$ and no power of Y is in I, so I is not primary. On the other hand $\sqrt{(X^n, X^{n-1}Y)} = \sqrt{X^{n-1}(X, Y)} = \sqrt{(X^{n-1})} \cap \sqrt{(X, Y)} = (X) \cap (X, Y) = (X)$. (Recall $\sqrt{I} = \cap P$ where P ranges over primes containing I.)

An ideal I of a ring R such that $\sqrt{I}$ is a prime ideal was called a quasi-primary ideal by Fuchs in [F1]. Before we start talking about primal ideals, it seems fair to introduce some of the terminology Fuchs used. This will help in understanding [F2] where Fuchs introduced the primal ideals and some of his later work. Call an element x prime to an ideal I of a ring R if $xy \in I$ implies that $y \in I$, for all $x, y \in R$. Next an ideal I of a ring R is called primal if the elements that are not prime to I form an ideal P called the adjoint ideal of I. It is easy to see that P is a prime ideal. Obviously "not prime to I" translates to "zero divisors mod I" and we conclude, as we did in HD1103, that I is primal if and only if R/I is a zeta ring with $Z(R/I) = P/I$ a prime ideal.

As we saw in HD1103 (Proposition H), if I is a primary ideal of a ring R then R/I is a zeta ring. So a primary ideal is primal. But a primal ideal is not necessarily primary. To see this we look at Example A again. We have seen that $I = (X^n, X^{n-1}Y)$ is not primary in $K[X, Y]$. Now note that $I = (X^n, X^{n-1}Y) = X^{n-1}(X, Y)$ and that X is a prime element contained in the prime ideal (X, Y). By Corollary L of HD1103, $X^{n-1}(X, Y)$ is a primal ideal of $K[X, Y]$.

Example A gives a primal ideal such that its radical is prime i.e. a primal ideal that is quasi primary. Must a primal ideal be quasi primary? The answer is no. We produce an example using Proposition K of HD1103.

Example B. For any integers $n > 0$ and r such that $1 < r \leq n$ the ideal $I = X_1^{a_1} X_2^{a_2} ... X_r^{a_r}(X_1, X_2, ..., X_n)$ is a primal ideal, in $K[X_1, X_2, ..., X_n]$, which is not quasi primary.

Illustration: Note that $R = K[X_1, X_2, ..., X_n]$ is a UFD and so every (nonzero) element of R is primal in Cohn's sense (cf HD1104). So $X = X_1^{a_1} X_2^{a_2} ... X_r^{a_r}$ is primal and every nonunit factor of $X_1^{a_1} X_2^{a_2} ... X_r^{a_r}$ is in the prime $P = (X_1, X_2, ..., X_n)$. Now by Proposition K of HD1103, XP is a primal ideal. That $\sqrt{I} = \bigcap_{i=1}^{i=r} (X_i)$ is direct and as $r > 1$ we conclude that $\sqrt{I}$ is not a prime.

If you want a simple example of a primal ideal that is not quasi primary you can take $I = XY(X, Y)$ in $K[X, Y]$, the same argument will work.

Next there's the question that I could not find the answer to while writing HD1103. That is if I is quasi primary, must I be primal? The answer here is a somewhat abstract form of the answer provided by Bruce Olberding, and it works for non-Prufer domains as well.

Example C. Let D be an integral domain with at least two maximal ideals. Let K be the quotient field of D, X an indeterminate over K and let $R = D + XK[X]$. Then XR is a quasi primary ideal that is not primal.

We first show that R/XR is not a zeta ring. For this note that for all nonzero nonunit $d \in D$, we have $d \in Z(R/XR)$. The reason is $d(X/d) = X$. Of course X/d is nonzero mod(XR) because $X \nmid X/d$ in R. Now because D has at least two maximal ideals we can find two nonzero nonunits $a, b \in D$ such that $a + b = 1$. So we have two zero divisors adding up to a nonzero divisor. This shows that $Z(R/XR)$ is not an ideal. Next to show that $\sqrt{XR}$ is a prime we need to make a few notes: (1) $XR = XD + X^2K[X] \subseteq XK[X]$ (2) If P is a prime ideal of $D + XK[X]$ then (a) $P = P \cap D + XK[X]$, if $P \cap D \neq (0)$ (b)$P = XK[X]$, with $P \cap D = (0)$ or (c) $P = (Xf(X) + 1)R$ with $P \cap D = (0)$, where $f(X) \in K[X]$, see e.g. [MZ] page 107 in a more general setting. It is easy to see that if $P \supseteq XR$ then $P = XK[X]$ or $P = P \cap D + XK[X]$ where $P \cap D \neq (0)$. So, $\sqrt{XR} = XK[X]$. This shows that XR is quasi primary but not primal.

To show that there are other ways to accomplish the task. I include below Bruce Olberding's example and his very apt illustration.

Example D. Let $R = Z + XQ[X]$, where Z = integers and Q = rational numbers. Then XR is a quasi-primary ideal since its radical is $XQ[X]$. I want to show that it is not primal, however.

Illustration: First, I want to claim that every maximal ideal contains an element that is prime to XR, since this will then disqualify every maximal ideal from being an adjoint prime of XR.

Let M be a maximal ideal of R containing XR. Then $M = pZ + XQ[X] = pR$ for some prime p. The claim is that p is prime to XR. For suppose $p(n + Xf(X)) = X(m + Xg(X))$ for some $n, m \in Z$ and $f(X), g(X) \in Q[X]$. Then $pn = 0$, so that $n = 0$. Hence $pXf(X) = mX + X\hat{}2g(X)$, so p divides m and $n + Xf(X) \in XR$. So p is prime to XR. So if XR is primal, then $XQ[X]$ must be the adjoint prime. To show this is not the case, it's enough to find an element of R that is not in $XQ[X]$ and is not prime to XR. Let n be a positive integer. Then $(n + X)((1/n)X) = X + (1/n)X^2 \in XR$, so $n + X$ is the desired element, and hence XR is not primal.

This example kind of hides the real issue. In a Prufer domain, an ideal is primal iff it is irreducible iff it is of the form $I = IR_M \cap R$ for some maximal ideal M of R. Using the latter fact, it's not hard to find lots of examples of quasiprimary ideals in Prufer domains that are not primal. Moreover, in an arbitrary domain, if I is primal, then $I = IR_P \cap R$ for the adjoint prime ideal P (this generalizes to non-domains too; in the notation of our papers, $I = I_{\{(P)\}}$), so similar ideas can be used to find quasi-primary principal ideals that are not primal in Noetherian rings. (By "our papers" he means the list provided in HD1103 of his papers with Heinzer and Fuchs.)

Another, more direct example comes from the use of formal power series.

Example E. Let D be an integral domain with at least two maximal ideals. Let K be the quotient field of D, X an indeterminate over K and let $R = D + XK[[X]]$. Then XR is a quasi primary ideal that is not primal.

Illustration. Indeed as $K[[X]]$ is a discrete valuation ring, $R = D + XK[[X]]$ is the classical $D+M$ construction (cf [BG]). From Theorem 2.1 of [BG] we can conclude that $M = XK[[X]]$ is a prime ideal of height one and that every prime ideal P of R with $P \neq M$ is of the form $p + XK[[X]]$ where p is a prime ideal of D. As before it is easy to see that $XR \subseteq XK[[X]]$. Thus $\sqrt{XR} = XK[[X]]$. That R/XR is not a zeta ring, can be shown as in Example C.

Now let us see why D having at least two maximal ideals is necessary. I will make another example of it.

Example F. Let D be a quasilocal domain with maximal ideal M, then in $R = D + XK[[X]]$ the ideal XR is quasi primary with $\sqrt{XR} = XK[[X]]$ and a primal ideal with adjoint $M + XK[[X]]$.

Illustration: That $XR \subseteq XK[[X]]$ is evident. Also the prime ideals Q of $D + XK[[X]]$ are of the form $Q = P + XK[[X]]$. This ensures that $\sqrt{XR} = XK[[X]]$. Next to see that XR is a primal ideal we let $a_1, a_2 \in Z(R/XR)$. Then $a_i = \alpha_i + t_i X f_i$ where $f_i \in R$, $\alpha_i \in M$, $t_i \in K$. If $\alpha_1 + \alpha_2 = 0$ then obviously $a_1 + a_2 = 0 \bmod (XR)$. If on the other hand $\alpha_1 + \alpha_2 \neq 0$ then we can write $a_1 + a_2 = \alpha_1 + \alpha_2 + t_1 X f_1 + t_2 X f_2 = (\alpha_1 + \alpha_2)(1 + tXf)$ where $f \in D + XK[[X]]$. Now as D is quasilocal and α_i nonunits we have $\alpha_1 + \alpha_2$ a nonunit and so $X/(\alpha_1 + \alpha_2) \notin XR$ such that $(\alpha_1 + \alpha_2)(1 + Xf)(X/(\alpha_1 + \alpha_2)) \in XR$. Hence the sum of every two zero divisors is a zero divisor. So R/XR is a zeta ring and XR is a primal ideal.

Tiberiu Dumitrescu observes that Examples C,D and F can be treated in one go. His treatment involves idealization. You may consult HD0904 for a brief introduction to idealization of a module.

Let D be a domain with quotient field K. Let R be either $D + XK[X]$ or $D + XK[[X]]$. Consider the ideal $I = XR$. Now the factor ring R/I is isomorphic to the ring $S = D(+)(K/D)$ (the idealization of the D-module K/D). Recall that the multiplication in S is given by $(a,p)(b,q) = (ab, aq + bp)$, (and addition is the usual coordinatewise addition). Now it suffices to compute the nilpotent elements and the zero-divisors of S.

The elements in K/D are nilpotent since their square is zero.

So, looking for other nilpotents means to consider elements $(a, 0)$ with $a \in D$. As $(a,0)^n = (a^n, 0) = (0,0)$ implies $a^n = 0$ implies $a = 0$, we have no other nilpotens. So the nilradical of S is K/D. This is a prime ideal of S because $S/(K/D) = D$. Thus I is quasi-primary.

Now let us look for the zero-divisors. As zero-divisor + nilpotent = zero-divisor, again we can confine our search to elements of type $(a, 0)$ with $a \in D$. If "a" is a unit in D, then $(a, 0)$ is a unit in S. If "a" is a nonunit in D, then $(a,0)(0, 1/a + D) = (0,0)$, so $(a,0)$ is a zero-divisor in S. Thus in S every element is a unit or a zero-divisor. Consequently, (0) is primal in S iff S is quasilocal. Finally, since K/D is consisting of nilpotents, we get that the spectral map $Spec(S)->Spec(D)$ is a homeomorphism. Thus I is primal in R iff (0) is primal in S iff S is quasi-local iff D is quasi-local.

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References:

[BG] E. Bastida and R. Gilmer, Overrings and divisorial ideals of rings of the form $D+M$" Michigan Math. J. 20(1973), 79-95.

[F1] L. Fuchs, On quasi primary ideals, Acta Univ. Szeged, vol. 11 (1947), 174-183.

[F2] L. Fuchs, On primal ideals. Proc. Amer. Math. Soc. 1 (1950), 1–6.

[MZ] J. Mott and M. Zafrullah, Unruly Hilbert domains, Canad Math. Bull. 33 (1990), 106-109.