QUESTION: (HD1202) Is there an easier method of finding an almost factorial domain that is not locally factorial

ANSWER. Yes there are some, but their being "easier" may be debatable. I give below 2 such examples, I and II

I. Look up Fossum's book [F]. Let's take the first paragraph of section 16 almost verbatim: Suppose that G is a finite group of automorphisms of a Krull domain B. Let A denote the fixed ring of B (i.e. $A = B^G$). The group acts on the quotient field L of B. Let $K = L^G$. Then it is easy to see that $K = qf(A)$. Since $A = B \cap K$, the ring A is also a Krull domain. Also as G is finite, B is integral over A.

Now consider the group $\{1, -1\}$ acting on $B = K[X, Y]$ with the action given by $X \mapsto -X$ and $Y \mapsto -Y$. Then $B^G = K[X^2, XY, Y^2] = A$ is a Krull domain, if *char* $K \neq 2$, and B is integral over A. Since B is integral over A every prime ideal of height one of B lies over a prime ideal of height one of A. David Anderson shows in [DFA] using some fairly advanced methods that $Cl(A) = Z/2Z$. Here's a somewhat simplified proof. (You may look up HD1105 for any concepts not introduced/explained here.)

Let $S = \{(XY)^n\}_{n=0}^{n=\infty}$ and consider A_S. Note that $A_S = K[X^2, XY, Y^2]_S \supseteq K[X/Y, Y^2] \supseteq K[X^2, XY, Y^2]$. So A_S is a quotient ring of $K[X/Y, Y^2]$. But as X/Y and Y^2 are algebraically independent over K, $K[X/Y, Y^2]$ is a UFD. But then, since A_S is a quotient ring of $K[X/Y, Y^2]$ we conclude that A_S is a UFD. Now by Nagata's Theorem, especially Corollary 7.2 of [F], $Cl(A) \to Cl(A_S)$ is a surjection and so every non-principal height one prime of A is in the class of a height one prime that contains XY. Let P be such a prime. Then as P is of height one a height one and hence principal prime $f(X, Y)B$ of B must lie over P. That is $XY \in f(X, Y)B \cap A$, where $f(X, Y)$ is a prime. So$XY = f(X, Y)g(X, Y)$ in B. As $f(X, Y)$ is a prime and $f(X, Y) \mid XY$ we must have $f(X, Y) = X$ or Y. This gives us two choices for primes containing XY : $P = XB \cap A = (X^2, XY)$ or $Q = YB \cap A = (XY, Y^2)$. Now $P = (X^2, XY) = \frac{X}{Y}(XY, Y^2) = \frac{X^2}{XY}(XY, Y^2) = \frac{X^2}{XY}Q$. Thus $[P] = [Q]$. So there is only one class to worry about. Next

$P^2 = (X^4,\ X^3Y,\ X^2Y^2) = X^2(X^2, XY, Y^2)$ and applying the v-operation we get $(P^2)_v = (X^2(X^2, XY, Y^2))_v = X^2(X^2, XY, Y^2)_v = (X^2)$. Thus $[P]^2 = [(P^2)_v] = [(X^2)] = 0$.

Thus $Cl(A) = \{[P] : [P]^2 = 0\}$ which is isomorphic to $Z/2Z$. To see that $K[X^2, XY, Y^2]$ is not locally factorial localize at the prime ideal (X^2, XY, Y^2) and note that X^2, XY, Y^2 are, each, irreducible in $K[X^2, XY, Y^2]_{(X^2, XY, Y^2)}$ and that they are not primes because $(X^2)(Y^2) = (XY)^2$.

We may note here that $P^2 = (X^4,\ X^3Y,\ X^2Y^2) = X^2(X^2, XY, Y^2)$ implies that $P^2 \subsetneq (X^2) = (P^2)_v$ as $1 \notin (X^2, XY, Y^2)$. So the prime ideals P and Q are other examples of divisorial ideals whose square is not divisorial.

Note 1. David Anderson has made the following observation: The only place you are using char $\neq 2$ is to get that $A = B^G = K[X^2, XY, Y^2]$, and thus A is a Krull domain. Note that for any field $K, A = K[X, Y] \cap K(XY, Y^2)$, and thus A is a Krull domain being the intersection of two Krull domains.

Note 2. Recently someone posed the following question: Let D be a locally factorial Krull domain with quotient field K and let L be a finite algebraic extension of K. Must the integral closure of D in L be locally factorial? The above example provides a simple answer in the negative.

Illustration: Note that for any field F the domain $F[X^2, Y^2]$ is a UFD and hence a locally factorial Krull domain. Also note that $D = F[X^2, Y^2] \subseteq F[X^2, XY, Y^2] = E$. Let $K = F(X^2, Y^2)$ and let $L = F(X^2, XY, Y^2)$. Since $(XY)^2 = X^2Y^2 \in K$ we conclude that $[L : K] = 2$. As E is a Krull domain, E is integrally closed. Also as E is integral over D, E is the integral closure of D in L. Now we already know that E is not locally factorial.

II. This example is based on a somewhat interesting polynomial ring construction: $D = R[X, r/X]$, where R is a commutative ring, though for our purposes R will be an integral domain with quotient field K, and $r \in R$. Note that $D = \oplus_{i\in Z} D_i$, a Z-graded ring where $D_n = X^n R$ and $D_{-n} = (r^n/X^n)R$ for $n \geq 0$. Also if $r \neq 0$ then $R[X, r/X] \cong R[S, T]/(ST - r)$ with $S \leftrightarrow X$ and $T \leftrightarrow r/X$. Obviously if $r = 0$ then $R[X, r/X] = R[X]$. If $r \notin U(R)$ the set of units of R then $U(R[X, r/X]) = U(R)$. Of course if $r \in U(R)$ then $R[X, r/X] = R[X, X^{-1}]$. The divisibility properties of these rings are studied by Dan Anderson and David Anderson in [AA]. (That is where our next example will come from.) Here's a list of some important observations and results from [AA].

(1). Suppose that $s \in R$ and $s \mid r$. Then $R[X, r/X]/(s, r/X) \cong R/(s)[X]$ and $R[X, r/X]/(s, X) \cong R/(s)[X^{-1}] \cong R/(s)[Y]$.

(2). $X^n R[X, r/X] \cap R = r^n R$ and $(r/X)^n R[X, r/X] \cap R = r^n R$.

(3). ([AA], Prop. 1) Let $0 \neq r$ and $D = R[X, r/X]$. Then the following are equivalent:

(i). X is irreducible (resp., prime)

(ii). r/X is irreducible (resp., prime)

(iii). $r \notin U(R)$ (resp., r is a prime in R).

(4). ([AA] Theorem 8 in part) $R[X, r/X]$ is integrally closed, is completely integrally closed, is a Krull domain if and only if R has the corresponding property.

(5). ([AA], Theorem 9) $R[X, r/X]$ is locally factorial if and only if R is locally factorial and for each maximal ideal M of R with $r \in M$, rR_M is a principal prime ideal.

(6). ([AA] Theorem 18 in part). Let R be a UFD, $0 \neq p \in R$ be prime, and $m \geq 1$. Then $Cl(R[X, p^m/X]) \cong Z/mZ$.

Example. Let R be a PID, $0 \neq p \in R$ be prime, and $m > 1$. Then $D = R[X, p^m/X]$ is an almost factorial domain that is not locally factorial.

Illustration: We shall repeat a part of the proof of Theorem 18, that is relevant, with some modifications. First off $R[X, p^m/X]$ is a Krull domain, by (4) above. Since R is a PID, $R[X, p^m/X][X^{-1}] = R[X, X^{-1}]$, is a UFD. Now by Nagata's Theorem, especially Corollary 7.2 of [F], $Cl(R[X, p^m/X]) \to Cl(R[X, p^m/X][X^{-1}])$ is a surjection and so every non-principal height one prime of A is in the class of a height one prime that contains X. Since $R[X, p^m/X]/(X)$

$\cong R/(p^m)[X^{-1}]$, one dimensional, the only minimal prime of $R[X, p^m/X]$ containing $XR[X, p^m/X]$ is (p, X). Next $(p, X)^m = (p^m, p^{m-1}X, p^{m-2}X^2, ..., pX^{m-1}, X^m) = (p^m/X, p^{m-1}, p^{m-2}X, ..., X^{m-1})X$. Let $H = (p^m/X, p^{m-1}, p^{m-2}X, ..., X^{m-1})$. We first show that $H_v = D$. To do this we show that H is contained in no divisorial prime ideal. Indeed let M be a prime ideal containing $H = (p^m/X, p^{m-1}, p^{m-2}X, ..., X^{m-1})$. Then M contains p^{m-1}, X^{m-1} and hence, p, X hence $M \supseteq (p, X, p^m/X)$. But as $R[X, p^m/X]$ is a Krull domain and (p, X) a height one prime ideal and hence a maximal divisorial ideal and $p^m/X \notin (p, X)$ we conclude that $M_v \supseteq (p, X, p^m/X)_v = D$. But as M was chosen arbitrarily, we conclude that $H_v = D$. Thus $((p, X)^m)_v = (HX)_v = (X)$. So $[(p, X)^m] = m[(p, X)] = 0$. In order to show that the order of $[(p, X)]$ is m assume that $0 < a < m$ is the order, then $((p, X)^a)_v = (f)$ for some $f \in D$. Then $m = ab$ for some integer b.This gives us $(X) = ((p, X)^m)_v = (f^b)$, so $b = 1$ because X is irreducible in D. Whence m is the order of $[(p, X)]$ and this gives $Cl(R[X, p^m/X]) = \{[(p, X)], [(p, X)^2], ..., [(p, X)^m]\} \cong Z/mZ$. Now we know that a Krull domain with torsion divisor class group is almost factorial and to see that D is not locally factorial we use (5) above or proceed as follows. Localize at $M = (X,\ p^m/X)$ and note that $X, p^m/X$ are both irreducible in D_M. Now $(X)(p^m/X) = p^m$ are two distinct factorizations. Hence D_M is not factorial.

Note. Having established that D is almost factorial and not locally factorial we conclude that the $(p, X)^m \subsetneq ((p, X)^m)_v$. For if not, then $(p, X)^m = (X)$, which makes (p, X) invertible and hence all the height one prime ideals in the class of (p, X) invertible. But by Nagata's theorem and Cor. 7.2 of [F] all nonprincipal height one primes are in the class of (p, X). This in turn makes D locally factorial, a contradiction. Whence $(p, X)^m = (p^m/X, p^{m-1}, p^{m-2}X, ..., X^{m-1})X \subsetneq (X) = ((p, X)^m)_v$.

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Muhammad Zafrullah

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