QUESTION: (HD 1205) You have shown that a PVMD is of finite t-character if and only if in it every nonzero t-locally principal ideal is t-invertible. Now given that the domain is t-locally a PVMD with every nonzero t-locally t-invertible t-ideal t-invertible, must the domain be of finite t-character using your result?

ANSWER: With just a little bit of insight, yes. Apparently you have seen a recent paper by Finocchiaro et al [FPT]. So I trust that you and the reader who got interested in the question are familiar with the concepts of v- and t-operations. (Those who aren't may look up my paper on "Putting t-invertibility to use" [Zt].) Before I get to the point let me make some introductory remarks. First let me note that I use D to denote a commutative integral domain, with quotient field K, and $F(D)$ to denote the set of nonzero fractional ideals of D. Let me also include the fact that D is called a Prufer v-multiplication domain (PVMD) if every nonzero finitely generated ideal of D is t-invertible. The result you alluded to appeared in [Zb] as Proposition 5, is given below.

Proposition A. A PVMD D is of finite t-character if and only if every t-locally principal t-ideal of D is t-invertible.

The following result was shown in [MMZ] as Corollary 1.8.

Proposition B. D is a PVMD if and only if for all $a, b \in D\backslash\{0\}$ the ideal $(a) \cap (b)$ is t-invertible.

When D is t-locally a PVMD with every nonzero t-locally t-invertible t-ideal t-invertible, we see that $(a) \cap (b)$ is t-locally t-invertible and so is t-invertible by the condition. This makes the domain into a PVMD and now you can apply the theorem you alluded to, i.e. Proposition A, noting that in a PVMD "t-locally t-invertible is t-invertible" is the same as "t-locally principal is t-invertible".

Remark X. In fact in all the situations where for each maximal t-ideal M of the domain D we have MD_M a t-ideal, t-locally t-invertible is t-invertible is the same as "t-locally principal is t-invertible".

Of course you have not asked for the converse, but let me refer you to Lemma 3 of [PVMD3] at http://www.lohar.com/researchpdf/PVMD3.pdf

The lemma referred to goes as follows.

"Lemma 3. (a). Let $D = \cap\{D_{S_\alpha}$ where S_α are multiplicative subsets of $D\}$. Let $a, b \in D$ such that $aD \cap bD$ is of finite type then $aD \cap bD$ is t-invertible if and only if $aD_{S_\alpha} \cap bD_{S_\alpha}$ is

t-invertible for each α.

(b) Let $D = \cap\{D_{S\alpha}$ where S_α are multiplicative subsets of $D\}$ and suppose that the intersection is locally finite. Let $a, b \in D$ such that $aD_{S_\alpha} \cap bD_{S_\alpha}$ is t-invertible for each α.

Then $aD \cap bD$ is t-invertible."

Part (b) of the above lemma applies to our situation once we set $S_\alpha = D\backslash P_\alpha$ where P_α ranges over maximal t-ideals of D. So if D is t-locally a PVMD and of finite t-character then D is a PVMD. Now you can apply Proposition A.

We have essentially proved the following result, using simpler techniques, and you can relate it to Proposition 2.8 of [FPT].

Proposition C. Let D be t-locally a PVMD, then D is a PVMD of finite t-character if and only if every t-locally t-invertible t-ideal is t-invertible.

Now to be fair, Proposition C is an improvement on the current knowledge and it can be put to several interesting uses, using my simple trick.

Of course what follows is not related in any way to your question but as I came across it answering your question I include it.

To see the uses note that a GCD domain is also a PVMD.

Corollary D. ([FPT] Corollary 2.9). Let D be t-locally a GCD domain, then D is a PVMD of finite t-character if and only if every t-locally principal ideal is t-invertible.

For the next corollary recall that an integral domain D is called a generalized GCD (GGCD) domain if for every pair $a, b \in D\backslash\{0\}$ $aD \cap bD$ is invertible. This concept was introduced in [AA] where it was also shown that in a GGCD domain every v-ideal of finite type is invertible. A GGCD domain is a PVMD, obviously.

Corollary E. Let D be locally a GCD domain. Then the following are equivalent.

(1) Every t-locally principal t-ideal is invertible
(2) Every t-locally principal t ideal is t-invertible.
(3) D is a GGCD domain of finite t-character.

Moreover if any of the above holds every nonzero locally principal ideal of D is invertible.

Proof. (1) $\Rightarrow$ (2) is obvious because invertible is t-invertible. For (2) $\Rightarrow$ (3) note that locally GCD implies t-locally GCD and so (2) makes D a PVMD of finite t-character. Now a locally GCD PVMD is GGCD [Zp, Cor. 3.4]. Now (3) $\Rightarrow$ (1) is straight-forward in that in a PVMD of finite t-character every t-locally principal ideal is t-invertible and a t-invertible t-ideal, being a v-ideal of finite type, in a GGCD domain is invertible.

(Alternatively, as one of my advisors points out, note that (3) $\Rightarrow$ (1) $\Rightarrow$ (2) hold with the assumptions and we can take proof of (2) $\Rightarrow$ (3) from above.)

The moreover part follows from Theorem 4 of [AZ], where a domain D is called an LPI-domain if every nonzero locally principal ideal is invertible. In Theorem 4 of [AZ] it is shown that if D is expressible as a locally finite intersection of localizations at prime ideals then D is an LPI-domain.

Before I give some more applications of my proof of Proposition C, let me show the validity of Remark X above. Note that by a t-local domain I mean a quasilocal domain whose maximal ideal is a t-ideal.

Proof of Remark X. The proof consists of noting that if (D, M) is a t-local domain then every t-invertible ideal of D is invertible and hence principal. Let A be a t-invertible ideal of D then for some fractional ideal B of D we have $(AB)_t = D$. That means that AB is in no maximal t-ideal and so AB is not contained in M the only maximal ideal of D. That is $AB = D$. This makes A invertible and hence principal. Now assume that in D, each nonzero t-locally principal t-ideal is t-invertible and consider a t-ideal A of D that is t-locally t-invertible. That is for an arbitrary maximal t-ideal M we have AD_M t-invertible. According to Remark X, MD_M is a t-ideal and hence by the starting remarks AD_M is principal. So A is t-invertible by the assumption. The converse is obvious because principal is invertible and hence t-invertible.

I called an integral domain D conditionally well behaved in [Zw] if for every maximal t-ideal M of D we have MD_M a t-ideal of D_M. Thus we have the following corollary.

Corollary F. Let D be a locally PVMD then the following are equivalent.

(1) D is conditionally well behaved and every nonzero t-locally principal t-ideal is t-invertible.

(2) Every nonzero t-locally t-invertible, t-ideal is t-invertible.

(3) D is a PVMD of finite t-character.

Moreover if D satisfies any of the above three conditions then D is an LPI-domain.

Proof. (2) $\Leftrightarrow$ (3) follows essentially from Proposition C with (or without) the proof given here. That (1) $\Rightarrow$ (2) follows from the proof of Remark X and (3) $\Rightarrow$ (1) because if D is a PVMD then for every prime t-ideal P of D we have PD_P a t-ideal, D_P being a valuation domain, and the rest follows from Proposition A. Again the "moreover" part follows from Theorem 4 of [AZ].

(Alternatively, as one of my advisors points out, note that (3) $\Rightarrow$ (1) $\Rightarrow$ (2) hold with the assumptions and we can take proof of (2) $\Rightarrow$ (3) from above.)

Corollary G. If D is a domain that is t-locally Krull, then the following are equivalent.

(1) D is conditionally well behaved and every t-locally principal t-ideal is t-invertible.

(2) Every prime t-ideal of D is of height one and every t-locally principal t-ideal is t-invertible.

(3) Every nonzero t-locally t-invertible, t-ideal is t-invertible.

(4) D is a Krull domain.

Moreover if D satisfies any of the above four conditions then D is an LPI-domain.

Proof. (1)$\Rightarrow$ (2) D being conditionally well behaved means that if M is a maximal t-ideal of D then MD_M is a maximal t-ideal of the Krull domain D_M. But that makes M of height one. (2)$\Rightarrow$ (1). Every prime t-ideal being of height one makes the domain automatically conditionally well behaved. (2)$\Rightarrow$ (4). Every maximal t-ideal M being of height one with D_M Krull makes D_M a one dimensional local Krull domain and hence a discrete rank one valuation domain. Now by Proposition A, D is of finite t-character and a locally finite intersection of discrete rank one valuation domains is a Krull domain. Now (4) implies every one of (1), (2) and (3) directly. (3)$\Rightarrow$ (4). By Proposition C D is a PVMD of finite t-character and a locally finite intersection of Krull domains is Krull.

(The above proof may be shortened. Anyone interested may shorten it or improve on it, "independently".)

References and Comments

[AA] D.D. Anderson and D.F. Anderson, “Generalized GCD domains” Comment. Math. Univ. St. Pauli 28(1979) 215- 221.

[AZ] D.D. Anderson and M. Zafrullah, Integral domains in which nonzero locally principal ideals are invertible, Comm. Algebra 39 (2011), no. 3, 933-941.

(This is an improved version of left-over notes from [Zb].)

[FPT] C.A. Finocchiaro, G. Picozza and F. Tartarone Star-Invertibility and t-finite character in Integral Domains, J. Algebra Appl., 10 (4) (2011) 755–769.

(This is a revised version of arxiv.org/pdf/1001.5220 which seemed uncannily similar to my paper with T. Dumitrescu: Characterizing domains of finite $*$-character [J. Pure Appl. Algebra, 214 (2010), 2087-2091] I am considering making public the referee report on my paper with Dumitrescu with my comments.)

[MMZ] S.B. Malik, J.L. Mott and Muhammad Zafrullah, On t-invertibility, Comm. Algebra 16(1988), 149-170.

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[Zt] __________, Putting t-invertibility to use, Non-Noetherian commutative ring theory, 429–457, Math. Appl., 520, Kluwer Acad. Publ., Dordrecht, 2000.

[PVMD3] Muhammad Zafrullah, Star operations of finite character induced by rings of fractions, available at http://www.lohar.com/researchpdf/PVMD3.pdf

(I sent this note to Marco Fontana in 2006. He did not show much interest in it, we got busy with other joint projects, and it got forgotten, sort of.) I have not checked how much of it is still "unpublished", but it might be interesting to go over it when you are reading about domains of finite t-character.

[Zb] __________, t-invertibility and Bazzoni-like statements J. Pure Appl. Algebra 214(2010), 654-657.