QUESTION: (HD 1209) Why is the $D+XD_S[X]$ construction from a GCD domain a Schreier domain? Could you give an example of a $D+XD_S[X]$ construction from a GCD domain D such that $D+XD_S[X]$ is not GCD?

ANSWER: There is an explanation given in [CMZ], just before Theorem 1.1, but perhaps you need a direct proof. For that we need to prepare a little, partly for you and partly for other readers.

Note that if D is an integral domain with S a multiplicative set in D and X an indeterminate over D_S, then the set $D+XD_S[X] = \{a_0 + \sum_{i=1}^{n} a_i X^i : a_0 \in D$ and $a_i \in D_S$ for $i \geq 1\} = \{f \in D_S[X] : f(0) \in D\}$ can be easily checked to be a ring and hence an integral domain [CMZ]. We know that if $\bar{S}$ is the saturation of S then $D_S = D_{\bar{S}}$. This gives $D+XD_S[X] = D+XD_{\bar{S}}[X]$ and so there is no harm in assuming that in $D+XD_S[X]$ the set S is saturated. Now note that for each $f = a_0 + \sum_{i=1}^{n} a_i X^i \in D+XD_S[X]$ we can find an $s \in S$ such that $f \in D[X/s]$. This gives $D+XD_S[X] \subseteq \bigcup_{s\in S} D[X/s]$. On the other hand for each $f \in \bigcup_{s\in S} D[X/s]$, $f \in D[X/s]$ for some $s \in S$. But as $f \in D[X/s]$ means $f = a_0 + \sum \frac{a_i}{s^i} X^i \in D+XD_S[X]$ and so $D+XD_S[X] = \bigcup_{s\in S} D[X/s]$. Next note that if there are $s,t \in S$ then there is $u = st$ say such that $D[X/s], D[X/t] \subseteq D[X/u]$. So $\{D[X/s]\}$ is a (an upper) directed set. Such a union of subrings is called a directed union or a direct limit. As a result of $\{D[X/s]\}$ being an upper directed set for any finite set of elements $a_0, a_1, \ldots, a_n \in D+XD_S[X]$ you can find an $s \in S$ such that $a_0, a_1, \ldots, a_n \in D[X/s]$.

Next an integral domain D is a pre-Schreier domain if D satisfies PS: for all $x,y,z \in D\backslash\{0\}$, $x \mid yz$ implies that $x = uv$ where $u \mid y$ and $v \mid z$. Pre-Schreier domains which were already known, without that name, were further studied in [Zp]. An integrally closed pre-Schreier domain is called a Schreier domain. Paul Cohn who introduced the notion of a Schreier domain in [C] had also shown that a GCD domain is a Schreier domain and that a polynomial ring over a Schreier domain is a Schreier domain. Now given that D is a GCD domain and S a saturated multiplicative set in D then $D+XD_S[X]$ is a directed union of the GCD domains $D[X/s]$, $s \in S$, each of which is a Schreier domain. Now let $x,y,z \in D+XD_S[X]$ with $x \mid yz$, say $yz = ax$. Then there is an $s \in S$ such that $a,x,y,z \in D[X/s]$ and the equation $yz = ax$ holds because it holds in $D+XD_S[X]$. But as $D[X/s]$ is a GCD domain, and hence a Schreier domain $x \mid yz$ in $D[X/s]$ implies that $x = uv$ where $u,v \in D[X/s]$ such that $u \mid y$ and $v \mid z$ in $D[X/s]$. Since elements of $D[X/s]$ are actually elements of $D+XD_S[X]$ we conclude that $u \mid y$ and $v \mid z$ in $D+XD_S[X]$. So $D+XD_S[X]$ is a pre-Schreier domain. To show that $D+XD_S[X]$ is integrally closed let's note that as $D+XD_S[X] \subseteq D_S[X]$, where $D_S[X]$ is integrally closed and has the same quotient field as $D+XD_S[X]$, it is enough to take $f \in D_S[X]$ and suppose that f is integral over $D+D_S[X]$. This means that f satisfies an equation of the form $f^n + a_{n-1}f^{n-1} + .. + a_1 f + a_0 = 0$ where $a_0, a_1, \ldots, a_{n-1} \in D+XD_S[X]$. By the directed union property there is an $s \in S$ such that $a_0, a_1, \ldots, a_{n-1} \in D[X/s]$. But then the whole equation $f^n + a_{n-1}f^{n-1} +..+a_1f + a_0 = 0$ is over $D[X/s]$ and $D[X/s]$ being integrally closed we get $f \in D[X/s] \subseteq D + XD_S[X]$ and so $D + XD_S[X]$ is integrally closed and so our proof that $D + XD_S[X]$ is a Schreier domain is complete. Now why can't $D + XD_S[X]$ be always a GCD domain? One answer is that GCD domains are not defined by first order statements and so a directed union of GCD domains may not be a GCD domain. The other answer is by providing a suitable simple example. A number of examples of non-GCD $D + XD_S[X]$ constructions from GCD domains are given in [Zgcd], but here is a particularly simple example which is easy to explain.

Example A. Let (V, M) be a discrete rank 2 valuation domain with minimal nonzero prime

ideal Q. Then $M = pV$ for a prime element p of V and $QV_Q = qV_Q$. Now consider the construction $R = V + XV_Q[X]$. I have taken this construction from section 2 of my paper [Zw]. As we have seen above $R = V + XV_Q[X]$ is Schreier. Now consider the pair q, X and note that every power of p divides q (because q belongs to Q) and every power of p divides X, by construction. Now let S be the multiplicative set generated by the prime p and consider $R_S = V_Q[X]$ which is a UFD and so q and X do not have a nonunit common factor in R_S. So, in R, the only common factors of q and X are powers of p. Now as for each power of p dividing both q and X there is a higher power, q and X have no GCD in R.

Now here is a natural question: When is a $D + XD_S[X]$ construction from a GCD domain a GCD domain? A simple answer was provided in Theorem 1.1 of [CMZ] and that is: $D + XD_S[X]$ is a GCD domain if and only if D is a GCD domain and GCD(d, X) exists for all $d \in D\backslash\{0\}$. There is an equivalent way of saying this is: $D + XD_S[X]$ is a GCD domain if and only if D is a GCD domain and S is a splitting multiplicative set of D (see [Zgcd]). Here a saturated multiplicative set S of D is a splitting multiplicative set if each $d \in D\backslash\{0\}$ can be written as $d = d_1 s$ such that $d_1D \cap Ds = dD$.

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References and Comments

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