QUESTION: (HD 1303) You showed that if A is a finitely generated ideal of a domain D and if S is a multiplicative set of D then $(AD_S)_v = (A_vD_S)_v$. Are there any examples where $A_vD_S \neq (AD_S)_v$?

ANSWER: First off A should be a finitely generated nonzero ideal and the result you mentioned appeared as part of Lemma 4 in [Zfc]. There are several ways of giving such examples. I will pick the example that is easy to see. I will then show you how to see the existence of such an example indirectly using some well known examples. As is apparent from the question the answer would involve star operations. You appear to have some idea but if some reader needs help I suggest looking up [G, sections 32 and 34].

For this direct example I would need a discrete rank 2 valuation domain V. A discrete rank two valuation domain is a valuation domain V with two nonzero prime ideals M the maximal ideal and P the height one prime ideal contained in M such that $M = mV$ a principal ideal and $PV_P = pV_P$ is a principal ideal. Now construct $R = V + XV_S[X]$, where $S = \{m^n : n \in N\}$. The ring R is a construction of the $D + XD_S[X]$ type that was studied in [CMZ] from where we can learn that $M + XV_S[X] \supseteq P + XV_S[X]$ are prime ideals of R and that $M + XV_S[X] = mR$. Since $P = \bigcap_{n=1}^{\infty} m^nV$ we conclude that $P + XV_S[X] = \bigcap_{n=1}^{\infty} m^nR$. Now $P + XV_S[X]$, being an intersection of principal ideals, is a v-ideal [G, 32.2] and hence a t-ideal.

Now note that $R_S = V_P[X]$ is a UFD, and that in a GCD domain a prime ideal that contains two coprime elements is not a prime t-ideal. Also note that for a, b in a GCD domain D, a, b are coprime if and only if $(a, b)_v = D$. It was shown in [Zgcd] that $(P + XV_S[X])R_S$ is not a prime t-ideal, because p and X are coprime in R_S. Our example here is just a modification.

Example A. Consider the ideal $A = (q, X)R$ where $q \in P\backslash\{0\}$. Of course $A \subseteq P + XV_S[X]$ and so $A_v \subseteq P + XV_S[X]$ which is a t-ideal. Thus we have $A_vR_S \subseteq (P + XV_S[X])R_S \subsetneq R_S$. On the other hand $(AR_S)_v = ((q, X)R_S)_v = R_S$ because in R_S, $q = p^r$ for some r and X are coprime, here p is the generator of the maximal ideal PV_P of V_P. So, while $(AR_S)_v = (A_vR_S)_v = R_S$ we have $A_vR_S \subsetneq (AR_S)_v$.

Remark B. (1) I used an often well understood convention in the expression $(AD_S)_v = (A_vD_S)_v$, the convention is: The v-operation is w.r.t the ring whose ideal it applies to. Let me explain: Let v_D be the v-operation on D and let v_{D_S} be the v-operation on D_S. Then the equation $(AD_S)_v = (A_vD_S)_v$ stands for $(AD_S)_{v_{D_S}} = (A_{v_D}D_S)_{v_{D_S}}$.

(2) The example above could be gleaned, easily, from Proposition 2.5 of [Zgcd]. But as you asked it seems pertinent to make public the answer to your question.

(3) Note that $A_vD_S \neq (AD_S)_v$ is equivalent to A_vD_S is not a v-ideal of D_S. For $AD_S \subseteq A_vD_S$ and if A_vD_S is divisorial then $(AD_S)_v \subseteq A_vD_S \subseteq (A_vD_S)_v$ which forces $A_vD_S = (AD_S)_v$ (because $(AD_S)_v = (A_vD_S)_v$).

(4) To see where else you can indirectly get an answer to your question look up section 4.2 of [Zgcd].

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