QUESTION: (HD 1402) Let D be a PVMD and a nonzero in D such that every nonzero x divides a power of a then is it true that the Krull dimension of D is equal

to the Krull dimension of the l-group of divisors of D ?

ANSWER: Let us simplify the question. A GCD domain D is a PVMD and in this case the group of divisors of D is just the group of divisibility $G(D)$ of D. Of course $G(D)$ is lattice ordered. The Krull dimension of a lattice ordered group G is, in view of comments in [M], just the prime filter dimension of G. This led Sheldon to study the prime filter (PF) primes in GCD domains in [S]. Briefly a prime ideal P is a PF-prime if for each pair $a,b \in P\backslash\{0\}$ we have $GCD(a,b) \in P$. The PF-dimension of D can be defined as the length of the longest chain of PF-primes. It turns out, via [M], that if D is a GCD domain then the PF-dimension of D is just the PF-dimension or the Krull dimension of $G(D)$. Now Sheldon [S] asked if there was a non-Bezout GCD domain D with PF-dimension $=$ Krull dimension. This question was answered in [CMZ] with an example of a non-Bezout GCD domain R such that Krull dim(R) $=$ PF-dim(R). Now you are, essentially, asking: Must there be a non-Prufer PVMD D with the added condition such that Krull-dim (D) $=$ Krull dim of the l-group of divisors of D? Turning it into the GCD question you are asking: Let D be a GCD domain and a nonzero in D such that every x divides a power of a then is it true that the Krull dimension of D is equal to the PF-dimension of D and D is not Bezout?

The answer is no. To see the answer let us collect the following observations. These observations will provide all the answers, hopefully.

Observation A. Let D be a GCD domain different from its quotient field K, X an indeterminate over K and let $R = D + XK[X]$ The the following hold.

(0) $D + XK[X]$ is a GCD domain.

(1) Let M be a prime ideal of R with $M \cap D = P \neq (0)$ then $M = P + XK[X]$ and M is a PF-prime of R if and only if P is a PF-prime of D.

(2) Let M be a prime ideal of R with $M \cap D = (0)$ then M is a height one principal prime ideal of the form $(1 + Xf(X))R$ where $(1 + Xf(X))$ is a prime in $K[X]$.

(3) If PF-dim(D) $< \infty$ then PF-dim(R) $=$ PF-dim(D)$+1$

(4) Krull dim(R) $=$ Krull dim(D)$+1$

Proof. (0) Because for each $d \in D\backslash\{0\}$ $GCD(d,X) = d$ we conclude by Theorem 1.1 of [CMZ] that R is a GCD domain.

(1) That $M = P + XK[X]$ follows from Theorem 4.21 of [CMZ] and for the other part we proceed as follows. Let $M = P + XK[X]$ be a PF-prime. Then for all $f,g \in M\backslash\{0\}$ $GCD(f,g) \in M$. In particular for all $a,b \in P\backslash\{0\}$ $GCD(a,b) \in M$ and as a,b are both of degree 0 in X so must be $GCD(a,b)$ whence for all $a,b \in P$, $GCD(a,b) \in P$ and that makes P a PF-prime. Conversely let P be a PF-prime of D and let $M = P + XK[X]$. Then as established by Sheldon [S] a prime ideal P in a GCD domain T is a PF-prime if and only if T_P is a valuation domain. Now we show that R_M is a valuation domain. Now $R_M \supseteq R_{D\backslash P} = D_P + XK[X]$, which is a Bezout domain by Corollary 4.13 of [CMZ], and a local overring of a Bezout domain is a valuation domain.

(2) Follows from Theorem 4.21 of [CMZ].

(3) Note that height $(P + XK[X])$ $=$ height $(P) + 1$.

(4) Follows from Corollary 2.10 of [CMZ].

Combining (0), (3) and (4) of Observation A we have the following observation.

Observation B. Let D be a GCD domain such that Krull dim (D) = PF-dim (D) and let K be the quotient field of D. If X is an indeterminate over K and $R = D + XK[X]$ then R is a GCD domain with Krull dim (R) = PF dim (R).

Observation C. There exists a GCD domain D with Krull dim (D) = PF dim (D)

Illustration: As shown in Example 3.1 of [CMZ], if D is a PID and S a multiplicative set of D with $D_S \neq K$ the quotient field of D and if X is an indeterminate then $R = D + XD_S[X]$ is a GCD domain with Krull dim (R) = PF dim (R).

As the PF-dimension is a specialization we have Krull dim $(D) \geq$ PF dim (D) for a GCD domain D. The examples of GCD domains D with Krull dim $(D) >$ PF dim (D) abound, for example every regular local ring of dimension greater than one is such an example. As a regular local ring R is a UFD its PF-dimension is one, which is less than the Krull dimension if the regular local ring is not a DVR.

Observation D. Let D be a regular local ring of dimension 2, $K = qf(D)$, X an indeterminate and $R = D + XK[X]$ then Krull dim $(R) = 3$ and PF-dim (R) $= 2$.

Now the problem with what we have done so far is that it does not contain a nonzero element a such that every nonzero element of divides a power of a. But this can be easily remedied.

Observation E. Let D be a GCD domain that is not equal to its quotient field K. Let X be an indeterminate and $R = D + XK[X]$. If $S = \{1 + Xf(X) : f(X) \in K[X]\}$ then (1) in R_S the element X is such that every nonzero element of R_S divides a power of X, (2) Krull dim (R_S) = Krull dim R and PF dim (R_S) = PF dim (R) (1) Since $R_S = (D + XK[X])_S = D + XK[X]_{(X)}$ every nonzero element of R_S is an associate of $d \in D\backslash(0)$ or an associate of an element of the form kX^r. Now d divides X and kX^r divides X^{r+1}. Thus every nonzero element of R_S divides some power of X. (We usually express this fact by saying that R_S is a G-domain.) (2) Since the Krull and PF-dimensions of R are governed by the heights of primes of the form $P + XK[X]$ with P a prime and the chains of those primes are disjoint with S we conclude that (2) holds.

Observation F. Let D be a GCD domain such that PF-dim (D) = Krull dim (D), $K = qf(D)$ and X an indeterminate. Then $R = D + XK[X]_{(X)}$ is a GCD domain with Krull dim (R) = PF dim (R) and an element X such that every nonzero element of R divides a power of X.

Observation G. Let D be a GCD domain such that PF dim (D) < Krull dim (D), $K = qf(D)$ and X an indeterminate. Then $R = D + XK[X]_{(X)}$ is a GCD domain with Krull dim $(R) >$ PF dim (R) and an element X such that every nonzero element of R divides a power of X.

References

[CMZ] D. Costa, J. Mott and M. Zafrullah, The Construction $D + XD_S[X]$, J. Algebra 53(2)(1978), 423-429.

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[S] P.B. Sheldon, Prime ideals in GCD domains, Canad J. Math. 26(1974), 98-107