QUESTION: (HD 1404) Let D be a domain and I be an ideal of D. Set $I^b = \cap IV_\alpha$ where the intersection is taken over all valuation overrings V_α of D. I know that $b : I \mapsto I^b$ is a star operation when D is integrally closed. But I do not know why b-operation is of finite type.

Answer: Note that a star operation $*$ is of finite type if $* = *_f$. That is for each $A \in F(D)$, the set of nonzero fractional ideals, $A^* = A^{*_f} = \cup\{F^* : F$ is a finitely generated nonzero subideal of $A\}$. Thus $*$ is of finite type if and only if for each $A \in F(D)$ the following holds: For each $x \in A^*$ there is a finitely generated subideal F of A such that $x \in F^*$. (Note that the t-operation is of finite type whereas the v-operation is not.

Example: Let (R,M) be a nondiscrete rank one valuation domain. Then $M^{-1} = R$ and so $M_v = R$ and there is $1 \in M_v$ such that $1 \notin (m_1,m_2,\ldots,m_n)_v$ for any nonzero $m_i \in M.$)

Before I give the answer, it might be helpful to include some pertinent information about Kronecker function rings. Of course you can find the information below from Gilmer's book [G], which is the main source on these topics in English. I am including it here in case you or some other readers do not have Gilmer's book handy.

(1) A star operation $*$ is called arithmetisch brauchbar (a.b.) if for all $A,B,C \in F(D)$, with A^*, $*$-finite $(AB)^* \subseteq (AC)^*$ implies $B^* \subseteq C^*$ and a star operation $*$ is called endlich arithmetisch brauchbar (e.a.b.) if for all $A,B,C \in F(D)$, with A^*,B^*,C^* all $*$-finite $(AB)^* \subseteq (AC)^*$ implies $B^* \subseteq C^*.$ It is easy to see that an a.b. star operation is also e.a.b.

(2) Two star operations $*_1, *_2$ are called equivalent if for all finitely generated $A \in F(D)$ we have $A^{*_1} = A^{*_2}$

(3) Theorem A. [G, Theorem 32.5]. Let D be an integral domain with quotient field L, and assume that $\{D_\alpha\}$ is a family of overrings of D such that $D = \cap D_\alpha$. If $F \in F(D)$, we define $F^* = \cap FD_\alpha$. Then the mapping $F \to F^*$ is a star operation $*$ on D and $FD_\alpha = F^*D_\alpha$. If each D_α is a valuation ring then this star operation $*$ is a.b. (and hence e.a.b.). (See [G, p.p. 396-397] for proof. Also note that (i) D_α is an overring of D means D_α contains D as a subring and $D \subseteq D_\alpha \subseteq qf(D)$, (ii) A family $\{D_\alpha\}$ of overrings of D such that $D = \cap D_\alpha$ is often called a defining family of $D.$ (iii) The operation $*$ described in Theorem A is said to be the operation "induced", on D, by the defining family $\{D_\alpha\}$ of $D.$)

(4) Recall that an integral domain D is integrally closed if and only if there is a family of valuation overrings $\{V_\alpha\}$ of D such that $D = \cap V_\alpha$.(A valuation domain is integrally closed and D is integrally closed if D is an intersection of integrally closed overrings. For the converse you may use [G, Theorem 19.8].)

(5) If D is integrally closed and if $\{V_\alpha\}$ is a family of valuation overrings of D, the star operation induced by $\{V_\alpha\}$ is called in [G] a w-operation. We may choose to call it an ω-operation as the term "w-operation" is more commonly used for another star operation studied in [Wang-Mc]. By Theorem A the ω-operation is an a.b. operation. The special ω-operation induced when $\{V_\alpha\}$ is the set of all valuation overrings of D is called a b operation, about which you have asked the question.

(6) Theorem B [G, Theorem 32.7] Suppose that $*$ is an e.a.b. operation and let $D^* = \{0\} \cup \{f/g : f,g \in D[X]\backslash\{0\}$ with $A_f^* \subseteq A_g^*\}$. (Here A_f denotes the content of f, the ideal generated by the coefficients of $f.$) Then the following hold.

(a) D^* is a domain with identity with quotient field $L(X)$, and $D^* \cap L = D.$ (Here

$L = qf(D)$.)

(b) D^* is a Bezout domain.

(c) If A is a finitely generated ideal of D, then $AD^* \cap L = A^*$.

(The ring D^* described in Theorem B is called the Kronecker function ring of D with respect to the e.a.b. operation $*$ and X, where X could represent a family of indeterminates over D, but for our purposes, X s a single vatiable is fine.)

(7) If D admits an e.a.b. star operation $*$ then D is integrally closed.

(8) ([G, Remark 32.9]) Two e.a.b. star operations $*_1, *_2$ are equivalent if and only if $D^{*_1} = D^{*_2}$.

(9) ([G, Theorem 32.10]) Let D be an integrally closed domain with quotient field L and let $*$ be an e.a.b. star operation on D, with Kronecker function ring D^*. If V^* is a valuation overring of D^*, then V^* is the trivial extension, to $L(X)$, of the valuation overring $V^* \cap L = V$ of D. (Here if V is a valuation overring of D then $V(X) = V[X]_S$ is called the trivial extension of V to $L(X)$, where $S = \{f \in V[X] : A_f = D\}$.)

(10) [G, Theorem 32.11] Let D be integrally closed with quotient field L, let $\{V_\alpha\}$ be a family of valuation overrings of D such that $D = \cap V_\alpha$ and let ω be the star operation induced by $\{V_\alpha\}$ on D. Then $D^\omega = \cap V_\alpha(X)$.

Now to see that $I \mapsto I^b$ is of finite type note that as b is induced by the set $\{V_\alpha\}$ of all valuation overrings of D we have $IV = I^b V_\alpha$ for each α. So we have $IV_\alpha(X) = I^b V_\alpha(X)$. Next as $D^b = \cap V_\alpha(X)$ and as D^b is Bezout we have that every ideal of D^b is complete. In particular ID^b is complete. Thus $ID^b = \cap IV_\alpha(X) = \cap I^b V_\alpha(X) \supseteq I^b D^b$. As $I \subseteq I^b$ we conclude that $ID^b = I^b D^b$. Now let $x \in I^b$. Then $x \in I^b D^b = ID^b$. So $x = \sum a_i f_i$ where $a_i \in I$ and $f_i \in D^b$. Thus $x \in FD^b$ where F is a finitely generated subideal of I. Next as $x \in I^b \subseteq K$ the quotient field of D we conclude that $x \in FD^b \cap K = F^b$ by Theorem 32.7 of [G].

[G] R. Gilmer, Multiplicative Ideal Theory, Marcel-Dekker, 1972.

[Wang-Mc] F. Wang and R. McCasland, On w-modules over strong Mori domains" Comm. Algebra 25(1997) 1285-1306.

Note: I sent Marco Fontana the above material and he responded with: "The fact that the b-operation is of finite type is discussed in details in

Fontana, Marco; Loper, K. Alan: Cancellation properties in ideal systems: a classification of e.a.b. semistar operations, J. Pure Appl. Algebra 213 (2009), 2095-2103

and previously in

Fontana, Marco; Loper, K. Alan: Kronecker function rings: a general approach. Ideal theoretic methods in commutative algebra (Columbia, MO, 1999), 189–205, Lecture Notes in Pure and Appl. Math., 220, Dekker, New York, 2001.

Concerning this subject, I asked many years ago to Robert Gilmer why he introduced eab operations, since Krull used only ab operations and the natural examples of eab operations were in fact ab operations. He said that eab was a technical property (arosen in the seminar with his students in Tahallasee), which is enough to prove all what we need. But apparently, he did not know any example of eab non-ab operations.

For this reason with Alan (and then with Alan and Matsuda) we published an example of eab non ab operation."

Indeed the material mentioned by Prof. Fontana can be very useful, as the star operations are closely related to semistar operations.