QUESTION: (HD 1405) Why is $R := Z + Q[X]$ an H-domain? I know that R is a $D + M$ construction
as follow

$$\begin{array}{ccc} R & \rightarrow & Q[X] \\ \downarrow & & \downarrow \\ Z & \rightarrow & Q \end{array}$$

Let P be a maximal t-ideal of R. When $P \not\subseteq M$, we have $P + M = R$. How to show that P is a v-ideal of R?

Answer: First note that $Z + Q[X] = Q[X]$, because $Z \subseteq Q$. Perhaps you meant $R = Z + XQ[X]$. The remainder of the answer will be assuming that you meant this. $Z + XQ[X]$ is a (generalized) $D + M$ construction because we have picked in the PID $S = Q[X] = Q + XQ[X]$ the maximal ideal $M = XQ[X]$ and picked Z as a subring of $Q = Q[X]/XQ[X]$. (You may look up Brewer and Rutter's [Michigan Math J. 23(1) (1976), 33-42.] to have an idea of what a general $D + M$ construction is.)

Note here that while M is a maximal ideal of $S = K + M$, M is not a maximal ideal of $D + M$ if D is not a field. So your contention " When $P \not\subseteq M$, we have $P + M = R$" is false. Prime ideals of rings of the form $D + XK[X]$, where $K = qf(D)$ are characterized in Theorem 4.21 of Costa, Mott and Zafrullah's ([CMZ] =) [J. Algebra 53(1978), 423-439].

Now let me answer " Why is $R = Z + XQ[X]$ an H-domain?". By [CMZ, Corollary 4.13] $R = Z + XQ[X]$ is a Bezout domain and so every maximal ideal of $R = Z + XQ[X]$ is a a maximal t-ideal. Next by [CMZ, Theorem 4.21] every maximal ideal of $Z + XQ[X]$ is of the form $M + XQ[X]$ where M is a maximal ideal of Z or of the form $f(X)R$ where $f(X)$ is irreducible in $Q[X]$ and $f(0) = 1$. Now as Z is a PID and so every maximal ideal of Z is of the form pZ where p is a prime element. Thus the maximal (t-) ideals of $R = Z + XQ[X]$ are of the types: $pZ + XQ[X] = pR$ or $f(X)R$ where $f(X)$ is irreducible in $Q[X]$ with $f(0) = 1$. So in sum every maximal t-ideal of $R = Z + XQ[X]$ is principal. Now as each principal (nonzero) ideal is divisorial we conclude that every maximal t-ideal of $R = Z + XQ[X]$ is divisorial. Now recall Proposition 2.4 of [Michigan Math. J. 35(2)(1988) 291–300] which says, in part, that a domain D is an H-domain if and only if every maximal t-ideal of D is divisorial.