QUESTION: (HD 1406) Question1: On the page no 4502, in the fifth line of the paragraph next to the definition

1.1, it is written that "Thus, in some sense, the equivalence relation measures how far an

HFD (resp. BFD, atomic domain) is from being a UFD (resp. FFD, CKD)." I could not understand in which sense the above statement is written.

Question 2: In the example 2.1 (b), it is given that "D is an - FFD if and only if D is FFD with U D finite." I could not get how the U D becomes finite. Also is it not true that an

- FFD becomes - UFD?

Answer: I'll look at those questions one by one below:

Question1: On the page no 4502, in the fifth line of the paragraph next to the definition 1.1, it is written that "Thus, in some sense, the equivalence relation measures how far an HFD (resp. BFD, atomic domain) is from being a UFD (resp. FFD, CKD)." I could not understand in which sense the above statement is written.

Answer: The clue is in:

"Let $\sim$ be the associate (equivalence) relation on $\mathcal{A}(D)$, i.e., $x \sim y$ for $x, y \in D$ if and only if $y = ux$ for some $u \in U(D)$. Then a $\sim$-UFD (resp., $\sim$-QFD, $\sim$-FFD, $\sim$-CKD) is just a UFD (resp., QFD, FFD, CKD). Also, if D is a $\approx$-UFD (resp., $\approx$-QFD, $\approx$-FFD, $\approx$-CKD) for some equivalence relation $\approx$ on $A(D)$, then D is an HFD (resp., atomic domain, BFD, atomic domain). Thus, in some sense, $\approx$ measures how far an HFD (resp., BFD, atomic domain) is from being a UFD (resp., FFD, CKD)." It says that if D is a $\approx$-UFD (resp., $\approx$-QFD, $\approx$-FFD, $\approx$-CKD) (with $\approx$ defined differently from $\sim$) for some equivalence relation $\approx$ on $\mathcal{A}(D)$, then D is an HFD (resp., atomic domain, BFD, atomic domain)" at least. So the definition of $\approx$ determines how far is D (which is an HFD) from being a UFD). If $\approx$ is closer to $\sim$ then the $\approx$ -UFD (which is an HFD) is closer to UFD. Same with the rest of concepts within brackets.

Question 2: In the example 2.1 (b), it is given that "D is an $=$ - FFD if and only if D is FFD with $U(D)$ finite." I could not get how the $U(D)$ becomes finite. Also is it not true that an $=$ - FFD becomes $=$ - UFD?

Answer: Let's look at "D is an $=$ - FFD if and only if D is FFD with U(D) finite." I could not get how the $U(D)$ becomes finite."

Note that a FFD is an atomic domain in which each nonzero nonunit element x has at most a finite number of atomic factors, upto associates. Now take an FFD, $D,$ with infinite $U(D)$ and define on $\mathcal{A}(D)$ $\approx$ by $x \approx y$ if and only if $x = y.$ By this associates cease to replace associates and so the FFD is not an $=$-FFD, because there are infinitely many atomic factors of a nonzero nonunit $x,$ under $\approx.$ So for a FFD to be $=$-FFD we need $U(D)$ finite.

Next let K be a finite field and let $D = K[X^2, X^3].$ Then it is easy to see that D is a FFD which is not an HFD and $U(D)$ is finite. Now define on $\mathcal{A}(D)$ $\approx$ by $x \approx y$ if and only if $x = y.$ Then for a nonzero nonunit x in D if a is an atomic divisor of x then for each $i \in U(D)$ ai is a distinct divisor of $x.$ So if we consider $D = K[X^2, X^3]$ as a $=$-FFD then the number of atomic factors of each nonzero nonunit gets multiplied by $|U(D)|.$ But still the number uf atomic divisors of x under $\approx$ is finite and so a FFD with $U(D)$ finite is a $=$-FFD. Now to see that an $=$-FFD may not be a $=$-UFD, note that as already observed a $=$-UFD has to be an HFD, while the $=$-FFD $K[X^2, X^3]$, with K finite, is not an HFD because $(X^2)^3 = (X^3)^2$.