QUESTION: (HD 1501) Let R be an integral domain and J be an ideal of R which is not contained in any non principal prime ideal of R. Is this ideal J principal? (Umar Nazir of Department of Mathematics, COMCASTS, Attok, Pakistan, asked this question.)

Answer: Yes the ideal J with that description would have to be principal. But before we prove it we need to note that each of the principal prime ideals containing J would have to be maximal, if it isn't it would have to be in some non-principal prime ideal as you cannot have a nonzero principal prime ideal properly inside a principal prime ideal.

Now suppose that J is such an ideal. If J is not contained in any principal prime then J is not contained in any prime, by its definition, and so $J = R$. So let J be a proper ideal, $A = \{p | \ pR \text{ is a prime ideal containing } J\}$ and S is a set multiplicatively generated by all primes in A. Two cases to consider: $S \cap J = \phi$ and $S \cap J \neq \phi$.

If $S \cap J = \phi$, then there is a prime ideal Q containing J such that $Q \cap S = \phi$. But then Q is not one of the principal primes containing J and so Q must be a non-principal prime a contradiction, showing that $S \cap J = \phi$ cannot happen. This leaves $S \cap J \neq \phi$. Let $x \in S \cap J$. Then for some set of primes $T = \{p_1, p_2, ...p_r\} \subseteq A$ we have $x = p_1^{n_1} p_2^{n_2} ... p_r^{n_r} \in J$. This means that if $pR \supseteq J$ then p must be one of the p_i. So, only a finite number of principal primes contain J. Let, after a rearrangement if necessary, $p_1, p_2, ..., p_s$ be all the primes such that $p_j R \supseteq J$. As for each $j = 1, ..., s$ only a finite power a_j of p_j is such that $p_j^{a_j} | x$ we can choose a_j to be maximal such. So that $J = p_1^{a_1} J_1$ where $J_1 \nsubseteqq p_1 R$, $J = p_1^{a_1} p_2^{a_2} J_2$ where J_2 is not contained in $p_1 R$ and $p_2 R$, and so on. In sum $J = p_1^{a_1} p_2^{a_2} ... p_s^{a_s} J_s$ where J_s is not contained in any $p_j R, j = 1, ..., s$. But these are all the primes containing J. So J_s is not contained in any principal maximal ideal because that would add to the set $\{p_1, p_2, ..., p_s\}$ and J_s cannot belong to a non-principal maximal ideal because of the condition on J. Thus J_s is in no maximal ideal and hence must be R. Thus $J = p_1^{a_1} p_2^{a_2} ... p_s^{a_s} R$.

Alternate Answer: A more sophisticated approach is suggested by Prof. Dan Anderson that uses his result: Let I be an ideal of a ring R with $I \neq R$. If every minimal prime ideal of I is finitely generated then I has only finitely many minimal primes. (See Theorem in [Proc. Amer. Math. Soc. 122(1)(1994), 13-14].)

Note that in this case every maximal principal ideal pR containing J is minimal over J too, for if Q were a minimal prime different from pR containing J, then Q would not be principal. Thus by Dan's theorem above J belongs to only finitely many principal primes. Let $p_1, p_2, ...p_s$ be all the primes such that $p_i R \sqsupseteq J$. Then as $p_1 R$ is minimal over J we have a positive integer a_1 such that $J = p_1^{a_1} J_1$ where $J_1 \nsubseteqq p_1 R$ (this is because if $J = p_1^n J_1$ for all n with $J_1 \subseteq p_1 R$ then pR would cease to be minimal over J as $\cap p_1^n R$ is a prime ideal) and as in the above proof we eventually have $J = p_1^{a_1} p_2^{a_2} ... p_s^{a_s} J_s$ where, as in the above proof, $J_s = R$.

Note 1. The specific description of J that Umar came across in his MS project (personal communication) seems to make the answer work. Thus if D is a domain with no non-maximal principal prime, such as a quasi-local ring with non-principal maximal ideal, then J with that specific description is the whole ring R. (This note was triggered by a comment from Dr. Shafiq ur Rehman, Umar's supervisor at COMCASRTS, Attok.)

Note 2. The ideal J must be different from (0) because even though in an integral domain the ideal (0) is a prime ideal it does not meet the definition of a principal prime, in that 0 is not regarded as a prime element and a principal prime should be generated by a prime.

Note 3. Prof. Dan Anderson took the time to write in hand an alternate solution to the problem, which is similar to what I have outlined, above, but more compact.

Suppose $0 \neq J \neq R$ is not contained in any non-principal primes. So all the primes minimal over J are principal. Thus by a result of prime (Dan's Theorem given above) there are only finitely many primes minimal over J, say $p_1R, p_2R, ...p_nR$. Now since $\cap_{n=1}^{\infty} p_i^n R$ is prime there exist s_i such that $J \subseteq p_i^{s_i}R$ but $J \nsubseteq p_i^{s_i+1}R$. So $J \subseteq p_1^{s_1}R \cap p_2^{s_2}R \cap ... \cap p_n^{s_n}R = p_1^{s_1}p_2^{s_2}...p_n^{s_n}R$. $J = (p_1^{s_1}p_2^{s_2}...p_n^{s_n})A$. Now A cannot be contained in a non-principal for then J would be. And certainly $J \nsubseteq p_iR$, so $A = R$, i.e. $J = p_1^{s_1}p_2^{s_2}...p_n^{s_n}R$.

Note 4. Prof. Dan Anderson has offered another alternate solution saying: Don't need a domain, slicker proof. The hypothesis gives every prime containing I is principal, so R/I has every prime ideal principal, so $R/I$is a PIR. Take a primary decomposition for I/I which lifts back to a primary decomposition for I. The primaries are clearly comaximal and each is a power of the prime and hence principal, so I is a product of principal prime ideals.

Note 5. Prof. G.M. Bergman has added this one liner for an alternate solution: Every nonprincipal ideal is contained in a maximal nonprincipal ideal, and every maximal nonprincipal ideal

is known to be prime. (For the first part of the sentence a simple Zorn's Lemma argument would suffice and for the second see Exercise 10 at page 8 of Kaplansky's Commutative Rings, Allyn and Bacon, 1970. The exercise is easy, if you follow the hint properly.)

Note 6. Any of the answers to the question can directly be modified to answer in the affirmative: If every nonzero prime ideal of a domain R is principal, must R be a PID?

Note 7: Profs. Said El-Baghdadi and G.M. Bergman have offered some interesting corrections and insights. I am thankful to all my helpers. I can see far because I stand on the shoulders of giants, with my little telescope.

N ote 8. It may be noted that the original, elementary answer, can be used to prove that every non-principal ideal I is contained in a prime ideal that is not principal, thus avoiding the somewhat involved proof of Exercise 10 at page 8 of Kaplansky's Commutative Rings, Allyn and Bacon, 1970. The argument goes as: Suppose that the non-principal ideal I is not contained in any non-principal prime ideals. But then by the above answer, I must be principal, a contradiction. This observation has been expanded to include various types of ideals in a recent write up:

http://www.lohar.com/researchpdf/Minimal%20primes%20of%20a%20star%20ideal4.pdf