QUESTION: (HD 1502) Is there an example of integral domain R, with fraction field K, such that, for some maximal ideal P of R, there exists a place of K extending the natural surjection from R to R/P, whose value field is a non trivial algebraic extension of R/P? This question was proposed by Michaël Bensimhoun, who also contributed with interesting remarks and examples.

ANSWER: We may need to prepare a little in order to make our examples easier to read. That means including a sort of working introduction to places and some terminology needed to explain the easy examples that we have in mind. Our treatment is more or less as in [B]. There are indeed different, equivalent, treatments available and those aren't too difficult to reconcile with ours.

Let K and L be two fields. Add ∞ to both K and L to form $\overline{K} = K \cup \{\infty\}$ and $\overline{L} = L \cup \{\infty\}$. Defining $a + \infty = \infty + a = \infty$ for all $a \in K, L$ and $a\infty = \infty a = \infty$ for all $a \in \overline{K}\backslash\{0\}, \overline{L}\backslash\{0\}$. (Note that $\infty + \infty, 0\infty$ and $\infty 0$ are not defined.) A function $f : \overline{K} \to \overline{L}$ is called a place on K w.r.t. L if (1) $f(x + y) = f(x) + f(y)$ for all $x, y \in \overline{K}$ for which $f(x) + f(y)$ is defined (2) $f(xy) = f(x)f(y)$ for all $x, y \in \overline{K}$ for which $f(x)f(y)$ is defined and (3) $f(1) = 1$.

Also note that if we assume $-\infty = \infty$, $0^{-1} = \infty$, $\infty^{-1} = 0$, we can extend the maps $x \to -x$ and $x \to x^{-1}$ to $\overline{K}$, $\overline{L}$ allowing $x + (-y) = x - y$.

Now as $\infty + \infty$ is not defined and so $f(\infty) + f(\infty)$ is not defined we must have $f(\infty) = \infty$........(1)

Further since $0.\infty$ and $f(0)f(\infty)$ we must have, as above $f(0) = 0$.........(2)

Next if $f(a)f(a^{-1})$ is defined then so is $aa^{-1} = 1$ defined but then $1 = f(1) = f(aa^{-1}) = f(a)f(a^{-1})$ which proves that

$f(a^{-1}) = (f(a))^{-1}$ for all $a \in \overline{K}$ ....(3)

Note that in case $f(a)f(a^{-1})$ is nor defined $f(a) = 0$ and $f(a^{-1}) = \infty$ or $f(a) = \infty$ and $f(a^{-1}) = 0$ and (3) still holds.

Next if $f(a) + f(-a)$ is defined then $a - a = 0$ is defined and so $0 = f(0) = f(a - a) = f(a) + f(-a)$ and this forces

$f(-a) = -f(a)$, for all $a \in \overline{K}$ ....(4)

Indeed if $f(a) + f(-a)$ is not defined then $f(a)$ must be ∞, forcing a to be ∞ and as we have agreed to put $-\infty = \infty$ we still have $f(-\infty) = -f(\infty)$.

From (3) and (4) we see that whenever $f(x) - f(y)$ and $f(x)f(y)^{-1}$ are defined we must have $f(x - y) = f(x) - f(y)$ and $f(xy^{-1}) = f(x)f(y)^{-1}$....... (5).

Next, for $x \in \overline{K}$, f is said to be finite at x if $f(x) \neq \infty$. By the above properties of f the set $E = \{f(x) : x \in \overline{K},\ f(x) \neq \infty\}$ is a subfield of $\overline{L}$. This set E is called the value field of f.

On the other hand the set $A = \{x \in \overline{K} : f(x) \neq \infty\}$ is a subring of $\overline{K}$, because if $x, y \in A$ then $x - y, xy \in A$ as $f(x) - f(y)$, $f(x)f(y) \neq \infty$ we have $\infty \neq f(x) - f(y) = f(x - y)$ and $\infty \neq f(x)f(y) = f(xy)$. That A is a valuation domain with quotient field K follows from the fact that if $x \in K\backslash A$ then $f(x^{-1}) = 1/f(x) = 1/\infty = 0$ that is for $x \in K$, $x \in A$ or $x^{-1} \in A$. It is easy to see that the set $m(A) = \{x \in K : f(x) = 0\}$ is the maximal ideal of A. The valuation ring A is also called the ring of f. Now note that if V is a valuation domain with quotient field K and residue field $\kappa(A)$ then we can define $f : \overline{K} \to \overline{\kappa(A)}$ by $f(x) = \infty$ if $x \notin A$ and show that f is a place with value field $\kappa(A)$. Summing it all up, and you may look up pages 381-385 of [B], we conclude that for each place f defined on $\overline{K}$ there is a valuation domain V with quotient field K such that the value field of f is $\kappa(A)$. This observation reduces your question to: Is there an example of an integral domain R, with fraction field K, such that, for some maximal ideal P of R, there exists a valuation overring V of R whose residue field is a non trivial algebraic extension of R/P?

In the following examples we will employ either both or the best possible of the above interpretations of the question.

Example 1. (a) Let $\mathbf{C}$ and $\mathbf{R}$ be the sets of complex and real numbers respectively.

Take $K = \mathbf{C}((X))$, $R = \mathbf{R} + X\mathbf{C}[[X]]$, it is easy to check that $\mathbf{C}((X)) = qf(R)$.

As R is quasi-local, the only choice for P is $P = X\mathbf{C}[[X]]$ and obviously $R/P = \mathbf{R}$.

Now define $f : \mathbf{C}((X)) \cup \{\infty\} \longrightarrow \mathbf{C} \cup \{\infty\}$ by: For all $u \in \mathbf{C}((X)) \cup \{\infty\}$, $f(u) = \infty$ if $u \notin \mathbf{C}[[X]]$ and $f(u) = u(0)$ the constant of u otherwise. It is easy to verify that f is indeed a place on $\mathbf{C}((X))$ with ring $\mathbf{C}[[X]]$ and that the "value field" of f is $\mathbf{C}$. Indeed $[\mathbf{C} : \mathbf{R}] = 2$.

Of course we can replace the extension of fields $\mathbf{R} \subset \mathbf{C}$, in the above example, by another extension $\mathbf{F} \subseteq \mathbf{L}$ of fields where $[\mathbf{L} : \mathbf{F}] = n < \infty$ to get examples of varying description as we see in the following example.

Example 1. (b) Let $\mathbf{F} \subseteq \mathbf{L}$ be an extension of fields with $[\mathbf{L} : \mathbf{F}] = n < \infty$. Set $R = \mathbf{F} + X\mathbf{L}[[X]]$ which is a local ring and note with maximal ideal $P = XL[[X]]$ and $R/P = \mathbf{F}$. Also note that $\mathbf{L}[[X]]$ is a valuation overring of R with maximal ideal $X\mathbf{L}[[X]]$ and $\kappa(\mathbf{L}[[X]]) = \mathbf{L}$ and $[\mathbf{L} : \mathbf{F}] = n$ our example meets the requirement.

As the power series constructions may be a bit hard to understand for some, we include an example using a similar polynomial construction.

Example 2. (a) Let $\mathbf{F} \subseteq \mathbf{L}$ be an extension of fields with $[\mathbf{L} : \mathbf{F}] = n < \infty$. Set $K = \mathbf{L}(X)$, $R = \mathbf{F} + X\mathbf{L}[X]$. Note that $\mathbf{L}(X) = qf(R)$ (Indeed $qf(R) = qf(R[X^{-1}]) = qf(\mathbf{L}[X, X^{-1}]) = qf(\mathbf{L}[X])$.) and the obvious choice of a maximal ideal P of R is $X\mathbf{L}[X]$. So $R/P = \mathbf{F}$. The choice of $P = X\mathbf{L}[X]$ suggests the choice of the valuation overring as $\mathbf{L}[X]_{(X)} = \mathbf{L} + X\mathbf{L}[X]_{(X)}$ and indeed $\kappa(\mathbf{L}[X]_{(X)}) = \mathbf{L}$.

Example 2. (b) Let $\mathbf{F} \subseteq \mathbf{L}$ be an extension of fields with $[\mathbf{L} : \mathbf{F}] = n < \infty$. Set $K = \mathbf{L}(X)$, $R = \mathbf{F} + X\mathbf{L}[X]$. Note that $\mathbf{L}(X) = qf(R)$ and the obvious choice of a maximal ideal P of R is $X\mathbf{L}[X]$. So $R/P = \mathbf{F}$. Now define $f : \overline{\mathbf{L}(X)} \to \overline{\mathbf{L}}$ by: For all $u \in \overline{\mathbf{L}(X)}$, $f(u) = \infty$ if $u \notin \mathbf{L}[X]_{(X)}$ and $f(u) = u(0)$ otherwise. Obviously f is a place with value field $\mathbf{L}$ and $[\mathbf{L} : \mathbf{F}] = n$. (The ring of f can be easily seen to be $L[X]_{(X)}$.)

Another example comes from $D + M$ constructions of the type studied in Bastida and Gilmer in [BG].

Example 3. Let $\mathbf{F} \subseteq \mathbf{L}$ be an extension of fields with $[\mathbf{L} : \mathbf{F}] = n < \infty$ and let V be a valuation domain of the form $V = \mathbf{L} + M$. Then the subring $R = \mathbf{F} + M$ is such that V/M is algebraic over R/M.

References

[A] S. Abhyankar, Local uniformization on algebraic surfaces over ground fields of characteristic $p \neq 0$, Annals of Mathematics 63(3) (1956), 491-526.

[B] N. Bourbaki, Elements of mathematics. Commutative algebra. Translated from the French. Hermann, Paris; Addison-Wesley Publishing Co., Reading, Mass., 1972.

[BG] E. Bastida and R. Gilmer, Michigan Math. J. 20 (1973), 79–95.

Note1: Examples 1 and 2 have been adapted from Examples 2, 3 on page 384 of [B]. Example 3 of [B] is more elaborate and I used the simpler version. But Michaël has pointed out that the full version of Example 3 of [B] could be used to provide examples of infinitely many valuation overrings V_α of $\mathbf{F}+X\mathbf{L}[X]$ with $\kappa(V_\alpha) = \mathbf{L}$. To construct V_α let $\alpha \in \mathbf{L}$ and note that $X - \alpha$ is a prime in the PID $\mathbf{L}[X]$. Let $P = (X - \alpha)$ and set $V_\alpha = \mathbf{L}[X]_P = \mathbf{L}[X]_{(X-\alpha)}$ which is a rank one discrete valuation ring with maximal ideal $P\mathbf{L}[X]_P$. Now $\kappa(V_\alpha) = \frac{\mathbf{L}[X]_P}{P\mathbf{L}[X]_P} \cong \frac{\mathbf{L}[X]}{P} = \frac{\mathbf{L}[X]}{(X-\alpha)} = \mathbf{L}[\alpha] = \mathbf{L}$.

Note2: Michaël has indeed raised the question of existence of V, R, P such that $\kappa(V)$ algebraic over R/P while R is integrally closed. An offhand answer to this question is: If there is such an example it cannot be of the form $V = \mathbf{L}[X]_{(X-\alpha)}$, $R = \mathbf{F} + X\mathbf{L}[X]$. For, as it is easy to see, $\mathbf{F} + X\mathbf{L}[X]$ is integrally closed if and only if $\mathbf{F}$ is algebraically closed in $\mathbf{L}$.

Note3: Tiberiu Dumitrescu has the following example for Michaël's question:

Examples of valuation overrings

The following example is inspired by the proof of Lemma 8 in S. Abhyankar's 1956 paper "On the valuations centered in a local domain", Amer. J. Math. 78(1956), 321-348. Consider the following commutative diagram

$$\begin{array}{ccccc}
\mathbb{Z}[X]_{(2,X+1)} & \to & W & \to & \mathbb{Q}[X]_{(X^2+3)} \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{Z}[\sqrt{-3}]_{(2,1+\sqrt{-3})} & \to & \mathbb{Z}[\frac{1+\sqrt{-3}}{2}]_{(2)} & \to & \mathbb{Q}(\sqrt{-3}) \\
\downarrow & & \downarrow & & \\
\mathbb{F}_2 & \to & \mathbb{F}_4. & &
\end{array}$$

The horizontal maps are inclusions, the first row of vertical maps is given by $X \mapsto \sqrt{-3}$ and the second row of vertical maps is given by $\sqrt{-3} \mapsto 1$. The rank-two valuation domain W is the pull-back of the right-up square (pull-back of two DVRs), $W = \{f(X) \in \mathbb{Q}[X]_{(X^2+3)} \mid f(\sqrt{-3}) \in \mathbb{Z}[\frac{1+\sqrt{-3}}{2}]_{(2)}\}$.

Thus we have a two-dimensional local regular domain with residue field $\mathbb{F}_2$ having a rank-two valuation overring with residue field $\mathbb{F}_4$.

Here is another example using the same pattern.

$$\begin{array}{ccccc}
\mathbb{Q}[X,Y]_{(X,Y)} & \to & V & \to & \mathbb{Q}[X,Y]_{(X^2+Y^2)} \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{Q}[T,iT]_{(T,iT)} & \to & \mathbb{Q}(i)[T]_{(T)} & \to & \mathbb{Q}(i,T) \\
\downarrow & & \downarrow & & \\
\mathbb{Q} & \to & \mathbb{Q}(i). & &
\end{array}$$

The horizontal maps are inclusions, the first row of vertical maps is given by $X \mapsto T, Y \mapsto iT$ and the second row of vertical maps is given by $T, iT \mapsto 0$. The rank-two valuation domain V is the pull-back of the right-up square (pull-back of two DVRs).

Note4: Actually Abhyankar's Lemma 8 of [A] amply answers the question, though of course examples help.

Lemma (Abhyankar's Lemma 8 of [A]. Let (R, M) be a two dimensional normal local domain with quotient field K . Let P be a minimal prime deal in R. Then: (1) R_p is the valuation ring of a real discrete valuation w of K; (2) there exists at least one and at most a finite number of valuations v of K having center in R which are composed with w, i. e., for which $R_v \subseteq R_w$; and (3) each such valuation v is discrete of rank two and R_v/M_v is a finite algebraic extension of R/M (hence in particular v is of R-dimension zero).

Note5. A number of people, other than Michaël, were exposed to earlier versions of this material, Professors Dobbs, Dumitrescu and Kabbaj responded, 3/22/2015 offering advice. I am grateful to them all and as usual, mistakes are all mine. Muhammad Zafrullah