QUESTION: (HD1503) Let $A \subseteq B$ be an extension of integral domains such that for all divisorial ideals I of A we have: (c_v): $I^{-1}B = (IB)^{-1}$. Is it equivalent to the condition (c) of your article [ABZ, J. Algebra Appl. 11 (2012), no. 1, 1250007, 18 pp]? if not, is the extension t-linked? (we can suppose that both A and B are Krull domains but neither A nor B is Dedekind, because if A is Dedekind or reflexif (c_v) = (c)) ((c): $I^{-1}B = (IB)^{-1}$ for all $I \in F(A)$). (Walid Maaref, a Tunisian student, asked this question.)

ANSWER: You are initially asking if the following two conditions are equivalent.

(c_v): $I^{-1}B = (IB)^{-1}$ for all divisorial ideals I of A and

(c): $I^{-1}B = (IB)^{-1}$ for all $I \in F(A)$, where $F(A)$ denotes the set of nonzero fractional ideals of A

The answer is no. To see the answer recall that if A is a UFD then I_v is principal for every nonzero ideal I of A.

Observation A: Let A be a UFD then for any extension $A \subseteq B$ of domains the condition (c_v) holds.

Proof. Let I be a divisorial ideal of A. Then $I = aA$ and $I^{-1} = \frac{1}{a}A$, $I^{-1}B = \frac{1}{a}AB = \frac{1}{a}B$, $(IB)^{-1} = (aAB)^{-1} = (aB)^{-1} = \frac{1}{a}B$. Thus showing that $I^{-1}B = (IB)^{-1}$ for all divisorial ideals I of A.

Next recall that in an extension of domains $A \subseteq B$ the ring B is said to be t-linked over A if $I^{-1} = A$ implies $(IB)^{-1} = B$ for each finitely generated ideal I of A.

Observation B. Let A be a UFD then A must be a PID to satisfy (c): $I^{-1}B$ $= (IB)^{-1}$ for all $I \in F(A)$ for any extension $A \subseteq B$ of domains.

Proof. If B is such that $I^{-1}B = (IB)^{-1}$ for all $I \in F(A)$ then, at least, B is t-linked over A. In particular, if condition (c) holds, every overring of A is t-linked over A. But then every maximal ideal of A is a t-ideal, by Theorem 2.6 of [DHLZ, Comm. Algebra 17(1989) 2835- 2852]. Now a UFD is Krull and hence a PVMD and a PVMD whose maximal ideals are t-ideals is Prufer (3(b) of Proposition 4.4 in [MZ, Manuscripta Math. 35(1981)1- 26]. Finally, a Prufer UFD is a PID.

Combining the above observations and noting that not all UFDs are PIDs we conclude that conditions (c) and (c_v) are not equivalent.

Next, sticking to the assumption that A is a UFD, and hence Krull we note that (c) and (c_v) are not equivalent when $A \subseteq B$ are Krull domains. This is because (c_v) would hold for any Krull domain B in $A \subseteq B$ while (c) would hold only if B, in $A \subseteq B$, is t-linked over A. From Observation A we can conclude that (c_v) on an extension $A \subseteq B$ may not force B to be t-linked over A.

Remark. We can take A in the above study to be a locally factorial Krull domain. For this all we have to do is note that if I is a nonzero ideal of a locally factorial Krull domain A then I_v is invertible (see e.g. (7) Theorem 3.1 of [Proc. Amer. Math. Soc. 85 (1982), 141-145]). Using the fact that if J is an invertible ideal of A then JB is an invertible ideal of B for every domain B containing A as a subring and $(JB)^{-1} = J^{-1}B$. Using this information we repeat Observations A and B as follows.

Observation A1: Let A be a locally factorial Krull domain then for any extension $A \subseteq B$ of domains the condition (c_v) holds.

Observation B1: Let A be a locally factorial Krull domain then A must be a Dedekind domain to satisfy (c): $I^{-1}B = (IB)^{-1}$ for all $I \in F(A)$ for any extension $A \subseteq B$ of domains.