QUESTION: (HD 1504) Did anyone ever look at domains with the property that if the gcd exists for a given pair, then the LCM exists for that given pair or if the gcd exists for a given pair it is a linear combination? This question was proposed by Professor Daniel Anderson.

ANSWER: I'll take parts of the question one by one.

The domains in which the following holds: "if the gcd exists for a given pair, then the LCM exists for that given pair".

It is patent that if for $a, b \in D\backslash\{0\}$ $LCM(a,b)$ exists then $GCD(a,b)$ exists. Thus we are looking at domains D in which $GCD(a,b) \Leftrightarrow LCM(a,b)$ exists.

It is easy to see that if $GCD(a,b)$ and $LCM(a,b)$ both exist then $LCM(a,b)D = \frac{ab}{GCD(a,b)}D = (a) \cap (b)$

Next if $GCD(a,b) = d$ then $a = a_1 d$ and $b = b_1 d$ where $GCD(a_1, b_1) = 1$. So, in these domains, $GCD(a,b) = d \Longleftrightarrow LCM(a,b) = a_1 b_1 d$, where a_1, b_1 are as described. Thus, in these domains, $GCD(x,y) = 1 \Longleftrightarrow LCM(x,y)D = xyD = (x) \cap (y) = xy(x,y)^{-1}$. Or, in these domains, $GCD(x,y) = 1 \Longleftrightarrow xyD = xy(x,y)^{-1}$. Cancelling xy we get $GCD(x,y) = 1 \Longleftrightarrow D = (x,y)^{-1}$ and as $(x,y)^{-1} = D \Longleftrightarrow ((x,y)^{-1})^{-1} = (x,y)_v = D$. When $(x,y)_v = D$ we say that x, y are v-coprime as we say that x, y are coprime when $GCD(x,y) = 1$. Thus in the domains in question any two coprime elements are v-coprime. Again if $GCD(a,b) = d$ then $a = a_1 d$ and $b = b_1 d$ where $GCD(a_1, b_1) = 1$ and so $(a,b)_v = d(a_1, b_1)_v = dD = GCD(a,b)D$ and $((a,b)_v)^{-1} = \frac{1}{ab}((a) \cap (b))$ and from $(a,b)_v = dD$ we get $((a,b)_v)^{-1} = (\frac{1}{d})$. Comparing, $\frac{1}{ab}((a) \cap (b)) = \frac{1}{d}D$ or $((a) \cap (b)) = \frac{ab}{d}D$. Thus a domain D in which $GCD(a,b)$ exists implies $LCM(a,b)$ exists is precisely the domain in which x, y coprime implies x, y v-coprime.

Now these domains do have a name! In [MZ, On Prufer v-multiplication domains, Manuscripta Math. 35(1981), 1-26], on page 18, a domain D is said to satisfy Property λ if any two coprime elements of D are v-coprime. The property appears to be quite toothless. But works wonders in the following situations.

(1) When D is atomic, i.e. every nonzero non unit of D is expressible as a finite product of irreducible elements.

Proposition 6.4 of [MZ] says: An atomic integral domain D is a UFD if and only if D satisfies the property λ.

In more general situations Corollary 6.5 of [MZ] says: If an integral domain D satisfies property λ then every atom of D is a prime.

(2) Of course every GCD domain satisfies property λ. But the property λ can be seen in a generalization of GCD domains, the so called pre-Schreier domains of [Z, Comm. Algebra 15(9) (1987), 1895-1920]. Using the proof of Lemma 2.1 of [Z1, J. Pure Appl. Algebra 65(1990) 199-207] we can establish that every pair of coprime elements of a pre-Schreier domain is v-coprime.

(3) Another generalization of GCD domains, the so-called Prufer v-multiplication domain PVMD does not generally satisfy the λ property. In fact, even a Prufer domain, a specialization of PVMDs, does not satisfy the λ property. This can be seen by taking a non-PID Dedekind domain D. Because D is not a PID, by

Proposition 6.4 of [MZ] D does not satisfy λ.

(4) Cohn [C, Bezout rings and their subrings, Proc. Cambridge Philos. Soc. 64 (1968), 251-264] called a domain D a pre-Bezout ring if for every pair $x, y \in D$, x, y coprime implies that x and y are comaximal. Now x, y being co-maximal means the GCD, 1, is a linear combination of x and y. And as $d = GCD(a, b) = dGCD(a_1, b_1)$ where a_1, b_1 are coprime, we conclude that pre-Bezout domains are precisely the domains in which the GCD of two elements a, b is a linear combination of a, b. (This much answers the part: if the gcd exists for a given pair it is a linear combination.) The pre-Bezout property was generalized to the GCD-Bezout property in [PT, Divisibility properties related to star operations on integral domains, Int. Electron. J. Algebra 12 (2012), 53-74] where Park and Tartarone study domains in which the GCD of a finite set of elements, if it exists, is a linear combination of those elements. Of interest to me is the fact that pre-Bezout and GCD-Bezout domains all satisfy the λ property.

That leaves: If LCM m of a, b exists when is m a linear combination of a, b? The answer, with a tongue in the cheek, is yes! Always. As we can always have $mD = a_1 b_1 d(1, x)$ for some x in D. But of course in the pre-Bezout domains case we can have $mD = a_1 b_1 d(a_1, b_1)$. In any case in the pre-Bezout domains this also is the case that if LCM of a, b exists, then GCD of a, b is a linear combination of a, b. Now note that, as we have already seen $(a) \cap (b)$ is principal if and only if $(a, b)_v$ is principal. Thus the domains in which LCM(a,b) exists implies GCD (a,b) is a linear combination of a,b are precisely the domains in which a, b v-coprime implies a, b co-maximal. These domains were discussed in [HZ, J. Algebra 423 (1)(2015) 93-113].

Comment added on 2-9-2020. About that Park-Tartarone paper on GCD-Bezout domains [Int. Electron. J. Algebra 12 (2012), 53-74]. I had a brief look into it again and realized that a so-called GCD-Bezout domain is nothing but the Special pre-Bezout domains of [DZ, J. Pure Appl. Algebra, 214 (2010), 2087-2091]. Let me elaborate on it. Indeed D may be assumed to be different from its field of quotients. Now reading the comments between Corollaries 11 and 12 of the DZ paper one gathers that D is a Special pre-Bezout (spre-Bezout) domain if and only if for every finite set of elements $x_1, ..., x_n$ in D the ideal $(x_1, ..., x_n)$ being primitive implies that $(x_1, ..., x_n) = D$. (Here $(x_1, ..., x_n)$ is primitive if $(x_1, ..., x_n) \subseteq xD$ implies that x is a unit.) On the other hand Park and Tartarone say that D is a GCD Bezout domain if whenever $GCD(x_1, ..., x_n) = d$ exists "we have a Bezout identity" which, in plain Math, means we have $d = (x_1, ..., x_n)$.

Now let's start. Spre-Bezout implies GCD-Bezout. Let d be a GCD of $(y_1, ..., y_n)$ and write $y_i = x_i d$. Then $(y_1, ..., y_n) = (x_1, ..., x_n)d$, where $(x_1, ..., x_n)$ is primitive because d is a GCD of the y_i. So, by the spre-Bezout property $(x_1, ..., x_n) = D$ forcing $(y_1, ..., y_n) = dD$ and this means that d is a linear combination of y_i or d satisfies the Bezout identity or whatever scholarly speak you want to speak. Conversely suppose that D is a GCD-Bezout domain and let $(x_1, ..., x_n)$ be a primitive ideal in D. Then 1 is a GCD of $(x_1, ..., x_n)$ and so by the GCD-Bezout property 1 is a linear combination of $x_1, ..., x_n$. That is $(x_1, ..., x_n) = D$. Oddly, after Corollary 12, the authors of DZ talk about the PSP property and lo and behold PSP property has good coverage in that Park-Tartarone paper.

Comment added on 8-24-2020. It is nice to see, at work, the honest tactic of changing the terminology and stealing from some friendless guys. This technique was however perfected by the great Korean Multiplicative Ideal Theorist, B.G. Kang, while working on his doctoral dissertation under the supervision of Professor D.D. Anderson. The target usually was my work or my work with those who either did not care or didn't have a voice.