QUESTION (HD 1505): Said El-Baghdadi and Hwankoo Kim ask, in a paper to appear in Communication in Algebra, if $D[[X]]$ is a generalized Krull domain when D is. Do you have any comments? I found the pre-print at ResearchGate under the title: Generalized Krull semigroup rings.

ANSWER: A Prufer v-Multiplication domain D is called a generalized Krull domain if for each maximal t-ideal M of D the localization D_M is a strongly discrete valuation domain as defined in [EKW] and every proper principal ideal has at most a finite number of minimal primes. It is well known that if D is a generalized Krull domain and S is a multiplicative set such that $D^{(S)} = D + XD_S[X]$ is a PVMD then $D^{(S)}$ is a generalized Krull domain [EGZ].

The answer to the question asked in [EK] appears to be NO as the following simple result indicates.

Proposition A. Let D be a PVMD such that there is a non-unit x with $\cap x^n D \neq (0)$ then the formal power series ring is not a PVMD.

Proof. Suppose that $D[[X]]$ is a PVMD. Then, in particular, $D[[X]]$ is integrally closed. But by Theorem 0.1 of [Ohm], $D[[X]]$ being integrally closed implies that D is integrally closed and for each non-unit x in D we have $\cap x^n D = (0)$. The presence of a non-unit x with $\cap x^n D \neq (0)$ will contradict this.

Proposition A may not appear to be conclusive but provides simple examples of generalized Krull domain counterexamples to the question.

Example B. Let D be a Krull domain that is different from its quotient field K. Then $R = D + YK[Y]$ is a generalized Krull domain such that $R[[X]]$ is not a generalized Krull domain.

Illustration: Let $d \in D$ be a nonzero non-unit. Then $0 \neq YK[Y] \subseteq \cap d^n R$ and by Proposition A, $R[[X]]$ is not a PVMD.

Conjecture C. Let D be a generalized Krull domain. Then $D[[X]]$ is a generalized Krull domain if and only if D is a Krull domain.

This conjecture will be verified if the following conjecture is verified. (G.W. Chang informs me that this conjecture can be verified using the fact that a generalized Krull domain is a t-SFT PVMD). (Said El-Baghdadi has the following verification. $D[[X]]$ generalized Krull domain implies D Krull.

Assume that D and $D[[X]]$ are generalized Krull. Then $D[[X]]$ is a PVMD because every generalized Krull domain is a PVMD. Also as D is generalized Krull every maximal t-ideal m of D is divisorial. This makes $m[[X]]$ a divisorial ideal by Kang and Anderson's result see Proposition 2.1 in Dobbs and Houston [DH]. Now a divisorial ideal is a t-ideal and as $D[[X]]$ being generalized Krull is a PVMD $D[[X]]_{m[[X]]}$ is a valuation domain. But then D_m is a rank one discrete valuation domain, by Theorem 1 of Arnold and Brewer [AB] and this holds for every maximal t-ideal m of D. But then $D = \bigcap_{m \in t-Max(D)} D_m$ is completely integrally closed and a completely integrally closed generalized Krull domain is a Krull domain.

Recall that an integral domain D is called an H-domain if for every ideal I, in $D, I^{-1} = D$ implies that there is a finitely generated $F \subseteq I$ such that $F^{-1} = D$. It was shown by Houston and Zafrullah [HZ] that D and H-domain if and only if every maximal t-ideal is divisorial. H-domains were introduced by Glaz and Vasconcelos in [GV] where it was shown that a completely integrally closed H-domain is a Krull domain. Indeed a Krull domain is an H-domain. With this introduction we state the following result.

Theorem C'. Let D be a PVMD that is also an H-domain. Then D is a Krull domain if and only if $D[[X]]$ is a PVMD.

Proof. If D is Krull, then it is well known that $D[[X]]$ is Krull and hence a PVMD. Conversely let D be an H-domain that is also a PVMD and let M be a maximal t-ideal of D and suppose that $D[[X]]$ is a PVMD. Then, as above, $M[[X]]$ is divisorial and hence a prime t-ideal of $D[[X]]$. But as $D[[X]]$ is a PVMD we conclude that $D[[X]]_{M[[X]}$ is a valuation domain. But as above, this forces D_M to be discrete rank one valuation domain and hence completely integrally closed. Now as M is an arbitrary maximal t-ideal of D and $D = \bigcap_{M\in t-Max(D)} D_M$ we conclude that D is a completely integrally closed H-domain and hence a Krull domain.

Of course as a generalized Krull domain is an H-domain. Theorem C' shows that the answer to the El Baghdadi-Kim question is, generally no.)

Conjecture D'. If D is a PVMD such that $D[[X]]$ is integrally closed then D is completely integrally closed.

Conjecture D' is sort of supported by the fact that if D is a GCD domain and $D[[X]]$ is integrally closed then D must be completely integrally closed. This follows from the fact that a GCD domain D is completely integrally closed if and only if for every non-unit x of D we have $\cap x^n D = (0)$.

The following result sets the stage for a possible resolution of conjecture D'. For better reading, however, we include some explanation of the notions mentioned in the proposition that follows.

A non-empty family S of nonzero ideals of a domain D is said to be a multiplicative system of ideals if $IJ \in S$, for each pair $I, J \in S$. If S is a multiplicative system, the set of ideals of D containing some ideal of S is still a multiplicative system, which is called the saturation of S and is denoted by $Sat(S)$. A multiplicative system S is said to be saturated if $S = Sat(S)$. If S is a multiplicative system of ideals, the overring $D_S := \cup\{(D : J); J \in S\}$ of D is called the generalized ring of fractions of D with respect to S. Indeed $D_S = S_{Sat(S)}$ and we choose $Sat(S) = \{J_v : \ J \in S\}$.

Proposition E. Let D be a PVMD. Then the complete integral closure D" of D is the generalized transform D_S where $S = \{I : I \subseteq D$ is t-invertible and $\cap(I^n)_v \neq (0)\}$ is a multiplicative set of ideals.

Proof. Let $x \in D_S$. Then $xI \subseteq D$ for some $I \in S$. Let $d \in \cap(I^n)v$. Then $dx^n \in D$ for all $n \geq 1$, and so$x \in D$".

Conversely suppose that $a/b \in D$". Then there exists $d \in D\backslash\{0\}$ such that $d(a/b)^n \in D$ for all $n \geq 1$. Thus $d \in (b^n) : (a^n)$ for all $n \geq 1$. But in a PVMD, $(b^n) : (a^n)$=$(((b) : (a))^n)_v$ for all $n \geq 1$ where $(b) : (a) \in S$ because $(b) : (a)$ is t-invertible and $0 \neq d \in \cap((((b) : (a)))^n)_v$ and so $a/b \in D_S$ because $a/b((b) : (a)) \subseteq D$.

Corollary F. A PVMD, D is completely integrally closed if for every proper t-invertible t-ideal I of D we have $\cap(I^n)_v = (0)$.

Proof. If D is completely integrally closed then $D" = D_S = D$ and this forces all non-trivial ideals I to be such that $\cap(I^n)_v = (0)$. The converse is obvious.

Corollary G. If a PVMD, D is completely integrally closed then for every non-unit r in D we have $\cap(r^n) = (0)$. (Indeed, as Tiberiu Dumitrescu pointed out, Corollary G is true for any completely integrally closed domain.)

Recall that D is a GGCD domain if every v-ideal of finite type of D is invertible. GGCD domains were studied in [AA] where it was shown that the complete integral closure of a GGCD domain D is an invertible generalized transform. Theorem E is a redo of that result. Because a GGCD domain is a PVMD in which every t-invertible t-deal is actually invertible Theorem 5 of [AA] and its corollary become special cases of Theorem E and Corollary F.

We would, of course like to resolve conjecture D'. One way of doing that would be to establish the connection, if it exists, between the events of the PVMD D being completely integrally closed with the event of its Kronecker function ring being completely integrally closed. For the Kronecker function ring is a Bezout domain which is known to be completely integrally closed if and only if for each non-unit r we have $\cap(r^n) = (0)$. In the absence of any insight in that direction we are reduced to making the best of the situation. Before we do that, let's note that a domain D is called Archimedean if $\cap(r^n) = (0)$ for each non-unit r of D.

Proposition H. Let D be a PVMD with the property that for every finite set $a_1, a_2, \ldots, a_s$ such that $(a_1, a_2, \ldots, a_s)_v \neq D$ there is a positive integer k such that a non unit $d \in D$ divides $(a_i)^k$ for each i. Then D is completely integrally closed if and only if D is Archimedean.

Proof. That a completely integrally closed PVMD is Archimedean comes free of charge, as shown in Corollary G. For the converse suppose that D is Archimedean, that is, $\cap(r^n) = (0)$ for each non-unit r of D. Now take a non-trivial t-invertible t-ideal I of D. Then $I = (a_1, a_2, \ldots, a_s)_v$ for some $a_1, a_2, \ldots, a_s \in D\backslash\{0\}$. Then, by the condition, there is a positive integer k, and a non-unit d such that

$(a_1^k, a_2^k, \ldots, a_s^k) \subseteq dD$ and this means that $(a_1^k, a_2^k, \ldots, a_s^k)_v \subseteq dD$. Now as D is a PVMD, we have $(a_1^k, a_2^k, \ldots, a_s^k)_v = ((a_1, a_2, \ldots, a_s)^k)_v$. Now I is t-invertible and$(I^k)_v \subseteq (d)$. Again, as we are working in a PVMD, $((I^{kt})_v \subseteq (d^t)$. Next, as for each positive integer n we have $n = kt + r$ where $t = 0, 1, 2, \ldots$ and $0 \leq r < k$ and as $I^{(kt+r)} \subseteq I^{kt}$ we conclude that for each positive integer n $(I^n)_v \subseteq (d^t)$ for a suitable t, where $t = 0$ when $n < k$ and $t = 1$ for $k \leq n$ $< 2k$...Setting $d^0 = D$ we have $\bigcap_{n=0}^{\infty}(I^n)_v \subseteq \bigcap_{n=0}^{\infty}(d^t)$. (We may reason thus: $\bigcap_{n=0}^{\infty}(I^n)_v \subseteq (I^n)_v \subseteq (d^t)$ when $n = kt + r$ where $t = 0, 1, 2, \ldots$ and $0 \leq r < k$,so $\bigcap_{n=0}^{\infty}(I^n)_v \subseteq (d^t)$ for $t = 0, 1, 2, ...$) But D is Archimedean, whence $\bigcap_{n=0}^{\infty}(I^n)_v = 0$

for each t-invertible t-ideal I of D, which forces D to be completely integrally closed.

Corollary K. Let D be a GCD domain then D is completely integrally closed if and only if D is Archimedean.

This is of course a known result, as pointed out in [AA]. Next, a domain with the QR property (every overring is a quotient ring) is known to be a Prufer domain such that for every nonzero ideal I there is $i \in I$ such that $I^n \subseteq (i)$. Thus we have the following corollary to Proposition H.

Corollary K1. A QR domain D is completely integrally closed if and only if D is Archimedean.

We can do somewhat better than Corollary K1. Recall that a domain D is called a t-QR domain if every t-linked overring of D is a quotient ring of D. Here in an extension $D \subseteq R \subseteq qf(D)$, R is said to be t-linked over D if for each finitely generated ideal A of D, $A^{-1} = D$ implies $(AR)^{-1} = D$. Obviously every flat overring is t-linked, so every overring of a Prufer domain is t-linked. In [DHLZ] it was shown that a PVMD D has the t-QR property if and only if for every nonzero finitely generated ideal I there is $b \in I_v$ such that $I^n \subseteq (b)$. Indeed as in a Prufer domain every nonzero finitely generated ideal is a v-ideal, this characterization reduces to that given by Pendleton in [P], of QR-domains. Thus we have the following result as well. (We are thankful to Tiberiu Dumitrescu for reminding us of [DHLZ].)

Corollary K2. A t-QR-PVMD D is completely integrally closed if and only if D is Archimedean. I am thankful to Tiberiu Dumitrescu for pointing out error in a statement about QR-domains. That led me to recall the t-QR property

For the next result recall that D is called an almost GCD (AGCD) domain if for every pair of nonzero elements $a, b \in D$ there is a positive integer n such that $(a^n) \cap (b^n)$ is principal. These domains were introduced in [Z] and further studied in [AZ]. It is well-known that D is an AGCD domain if and only if for every set of nonzero elements $a_1, a_2, ...a_s$ there is a positive integer k such that $(a_1^k, a_2^k, \ldots, a_s^k)_v = (d)$ for some $d \in D$. With this definition Theorem H clearly points to the following result.

Corollary L. An AGCD domain D is completely integrally closed if and only if D is Archimedean.

Recall that the set $t - inv(D)$ of t-invertible t-ideals of D is a group under t-multiplication. The set $P(D)$ of principal fractional ideals of D is clearly a subgroup of $t - inv(D)$. The quotient group $t - inv(D)/P(D)$ is called the class group or the t-class group of D and is usually denoted by $Cl_t(D)$. $Cl_t(D)$ was introduced in [B] where it was pointed out that $Cl_t(D)$ is the divisor class group if D is Krull and $Cl_t(D)$ is the ideal class group if D is Prufer. Also, it was shown in [Z] that an integrally closed AGCD domain is a PVMD with torsion t-class group and that a PVMD with torsion t-class group is an AGCD domain. Also as a Prufer domain is a PVMD too, a Prufer domain with a torsion class group is an AGCD domain. Thus a Prufer domain D with torsion class group is completely integrally closed if and only if D is Archimedean. Now a domain with the QR property is known to be a Prufer domain with torsion class group. Of course this fact is getting mentioned here because it was proved in [Ohm].

In the absence of a clear answer for PVMDs in general we look for the special cases. One special case is when a PVMD is a unique representation domain (URD), i.e. each proper principal ideal has finitely many minimal primes. It is well known that in a PVMD a minimal prime P of a principal ideal xD is a prime t-ideal and so D_P is a valuation domain.

Lemma M. Let a be a nonzero non-unit of a PVMD D, let P be a minimal prime of aD and let Q be a nonzero minimal prime of $\cap a^i D$. Then for every maximal prime t-ideal M_α containing Q we have $QD_{M_\alpha} = \cap a^i D_{M_\alpha}$.

Proof. Note that the prime ideal $\cap a^i D_{M_\alpha}$ sits right under the minimal prime of aD_{M_α} and so right under the prime ideal PD_{M_α}. But QD_{M_α} is the prime ideal that sits right under PD_{M_α}, Q being a minimal prime of $\cap a^i D$.

Noting that by Corollary 1.2 of [Ohm] $\cap a^i D$ is a radical ideal we conclude that the following holds.

Lemma N. Let D be a PVMD and let aD have only a finite number of minimal prime ideals $P_1, P_2, \ldots, P_n$. Then $\cap a^i D = Q_1 \cap Q_2 \cap \ldots \cap Q_n$ where each Q_i sits right under each P_i..

Recall that by Corollary 1.4 of [Ohm] if a is in a height one prime ideal then $\cap a^i D = (0)$. Thus we have the following conclusion.

Lemma P. Let D be a PVMD and let aD have only a finite number of minimal prime ideals $P_1, P_2, \ldots, P_n$ such that none of the P_i is of height one then $\cap a^i D \neq (0)$.

Note that in a PVMD a t-invertible t-ideal is t-locally principal and so we can treat it in the same manner as a principal ideal, more or less. Thus we have

Lemma Q. Let D be a PVMD and let A be a t-invertible t-ideal of D that has only a finite number of minimal prime ideals $P_1, P_2, \ldots, P_n$ such that none of the P_i is of height one then $\cap (A^i)_v \neq (0)$.

Recall also that, in a PVMD D, a t-invertible t-ideal A with a unique minimal prime is called a packet and naturally if the minimal prime of A is of height one then $\cap (A^i)_v = (0)$.

These results now lead to the following result.

Theorem R. Let D be a GCD URD and let X be an indeterminate over D. Then the following are equivalent.

(a) D is an intersection of rank 1 valuation rings,

(b) D is completely integrally closed,

(c) $D[[x]]$ is integrally closed,

(d) D is integrally closed, and $\cap a^i D = 0$ for every nonunit a in D.

(e) D is integrally closed and every nonunit of D is in a height one prime ideal.

(f) D is a Generalized UFD.

Proof. That (a)$\Rightarrow$ (b)$\Rightarrow$ (c)$\Rightarrow$ (d) was established in [Ohm].

(d)$\Rightarrow$ (e). By Lemma N, $\cap a^i D = Q_1 \cap Q_2 \cap \ldots \cap Q_n$ where Q_i are minimal primes of $\cap a^i D$. So

$\cap a^i D = 0$ implies that $Q_1 \cap Q_2 \cap \ldots \cap Q_n = (0)$ and this means at least one of Q_i say Q_1 is zero and this forces P_1 the corresponding minimal prime of aD to be of height one, by Lemma Q.

(e)⇒(f) . In a GCD URD, every nonzero non-unit a is expressible as a finite product of mutually co-prime packets, i.e. elements with unique minimal primes. Say $a = p_1 p_2 \dots p_n$ where each p_i has a unique minimal prime P_i, which by (e) is of height one. But then D_{P_i} is a rank one valuation domain, making p_i a rigid element such that for every nonunit factor r of p_i, p_i divides r^n for some n. This makes each packet a prime quantum and D a GUFD, as described in [AAZ] and in Zafrullah's doctoral dissertation [Zd].

(f) ⇒(a). As shown in [AAZ], a GUFD is a Generalized Krull domain of Ribenboim and so is a locally finite intersection of rank one (essential) valuations.

Ohm proved the equivalence of (a) through (e) for finite intersections of valuations.

Corollary S. A GCD ring of finite t-character is a GUFD if and only if $D[[X]]$ is integrally closed.

Of course the result that I set out to prove still eludes me and I still cannot show that if D is a PVMD URD and $D[[X]]$ integrally closed then D must be completely integrally closed.

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