QUESTION (HD 1601): Is there a Noetherian domain on which the star operations t, w, and d are distinct from one another?

ANSWER: For the star operations t, w, and d you may consult "Putting t-invertibility to use" in [Non-Noetherian commutative ring theory, 429–457, Math. Appl., 520, Kluwer Acad. Publ., Dordrecht, 2000]. I will assume basic knowledge of these operations. The example that comes to my mind right away is due to Abdeslam Mimouni. This example was cited in my paper with Anderson and El-Baghdadi [J. Algebra and Applications, 11(1) (2012) 1250007 (18 pages)] as Example 2.5. But this example has a problem. In this example $d = w$. We show that this problem can be easily fixed by going to the ring of polynomials. But first the example.

Example A. Let $R = Q(\sqrt{2})[[X, Y, Z]]$, where X, Y, Z are indeterminates over Q. Then $R = Q(\sqrt{2}) + M$ is a 3-dimensional integrally closed local (Noetherian) domain with maximal ideal $M = (X, Y, Z)R$. Now set $D = Q + M$. Then $D = Q + M$ is a local (Noetherian) domain with integral closure R (see [Brewer and Rutter, Michigan Math. J. 23 (1976) 33–42]). Since the maximal ideal M is common to both D and R, we have $M = MR$; and so for the prime ideals $P_1 = XR, P_2 = (X, Y)R$ of R, we have $P_1 \subsetneq P_2 \subsetneq M$. We claim that P_2 is not a t-ideal of D, while M is a t-ideal of D. This follows from the following observations. Since $ht_R(P_2) = 2$, we have $R = R : P_2 = (P_2 : P_2) = D : P_2$. Similarly, $R = R : M = M : M = D : M$. Now, as $M^{-1} = D : MD$, we must have $M_v \subsetneq D$. But since $D = Q + M$ is local, $M_v = M$. Next, since $R = D : P_2 = P_2^{-1} = M^{-1}$, we have $(P_2)_t = (P_2)_v = M_v = M$. But as $P_2 \subsetneq M$, we conclude that P_2 is not a t-ideal of D. Obviously P_2 is a w-ideal because, M being a t-ideal of D, every ideal of D is a w-ideal. That is the Achilles heel of this Noetherian ring D, while it has $d \neq t$ and $t \neq w$ it does have $d = w$. To tentatively correct this problem recall Proposition 2.6 from Mimouni's [Commun. Algebra 33 (2005) 1345–1355]. The Proposition says that if T is an indeterminate over D then $D[T]$ is a DW-domain (i.e. in D we have $d = w$) if and only if D is a field. Now as D, in our case, is patently not a field we conclude that $d \neq w$ in $D[T]$. But then we must show that $d \neq t$ and $t \neq w$. Obviously not every ideal of $D[T]$ is a t-ideal, for example (a, T) is not a t-ideal. So $d \neq t$.

We plan to show that $t \neq w$ by using the information we have on D. Our plan is based on the fact that for any nonzero ideal I of D we have $(I[T])^w = I^w[T]$, where T is an indeterminate thus if I is a w-ideal of D then $I[T]$ is a w-ideal of $D[T]$. To see this note that Hedstrom and Houston [HH, J. Pure Appl. Algebra 18(1980) 37-44.] studied the w-operation calling it an F_∞-operation. Hedstrom and Houston show that if $*$ is the F_∞-operation and I an ideal of D then $(I[T])^* = I^*[T]$ in Proposition 4.3 of [HH]. Now going back to the example in hand (Example A) we know that P_2 is a w-ideal of D and so $P_2[T]$ is a w-ideal of $D[T]$. Now, appealing to Proposition 4.3 of [HH] again, we conclude that $(P_2[T])_v = (P_2)_v[T] = M[T] \neq P_2[T]$. Now as D and $D[T]$ are both Noetherian $(P_2[T])_v = (P_2)_v[T]$ is the same as $(P_2[T])_t = (P_2)_t[T]$. Thus $t \neq w$ in $D[T]$.

Using the above reasoning, however, one can prove the following proposition.

Proposition B. Let D be a domain that is not a field and X an indeterminate over D. If $t \neq w$ over D then d, w and t are all distinct over $D[X]$.

Now the question is: Is the domain of the above example the only Noetherian domain to have a w-ideal that is not a t-ideal? The answer is, hardly.

Proposition C. Let D be a Noetherian local domain with maximal ideal a t-ideal. If $\dim(D) > 1$ then $t \neq w$ over D.

Proof. We note that because D is local with maximal ideal a t-ideal, every ideal of D is a w-ideal that is, $d = w$. Assume that, over D, $t = w$ also. Then as D is Noetherian $t = v$ already. So, D is a w-divisorial domain. But then D is one dimensional, by Theorem 4.2 of El-Baghdadi and Gabelli's [J. Algebra 285 (1) (2005) 335–355].

Note (1): Evan Houston has kindly pointed to another example and this example is one- dimensional and hence a "smaller" example: Let y denote the cube root of 2. Then $D = Q + XQ(y)[[X]]$ can be shown to be Noetherian local, as in Example A, above. That D is one-dimensional can be seen via results in Bastida and Gilmer's [Michigan Math. J. 20(1973) 79-95]. The ideal $(X, yX) \subsetneq XQ(y)[[X]]$ obviously. Next (X, yX) is not divisorial since $(D + Dy)_v = Q(y)[[X]]$, and so $(X, yX)_v = XQ(y)[[X]]$. As above, every ideal of D is a w-ideal, so $w = d$. Now D being Noetherian, we have $t = v$. But there is an ideal, (X, yX), that is not a v-ideal and hence not a t-ideal. Thus in D, $t \neq w$. Now we can add an indeterminate to complete the argument, as above. So, in sum, $(Q + XQ(y)[[X]])[T]$ is our example.

Note (2): Now Proposition B above indicates that the ring D with $t \neq w$ does not have to be local for the adjunction of an indeterminate to give the result. So why not take $D = Q + XQ(y)[X]$, where y is the cube root of 2? Indeed a look at page 114 of, Anderson, Anderson and Zafrullah's [Houston J. Math. 17(1991), 109-129] tells us that D is Noetherian and one dimensional. Also that every prime ideal of D different from $XQ(y)[X]$ is principal. That $XQ(y)[X]$ is divisorial follows from the fact that D is one dimensional Noetherian. So every maximal ideal is a t-ideal and $d = w$ in D. That $t \neq w$ can be established as above by noting that $(X, yX) \subsetneq XQ(y)[X]$, yet $(X, yX)_v = XQ(y)[X]$. Finally, by Proposition B $D[T]$ is such that d, w, t are all distinct.

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