QUESTION (HD 1701): Do star operations have any applications?

ANSWER: Recently, I wrote up the following. This might answer your question and hopefully more.

The questions: "When is an integral domain D a t-local domain?" and, "What good is a t-local domain?" may sound like the oddest questions. The simple answers to these questions are, "When D is a quasi-local domain and the maximal ideal of D is a t-ideal" and, "There are situations where the knowledge that a certain (quasi local) domain is t-local can simplify matters a great deal". The purpose of this note is to point out some telltale signs that would point to the fact that the domain is t-local and in some cases more. Usually, t-local domains being cousins of valuation domains, albeit distant ones, it helps to know the circumstances under which the knowledge that a quasi-local domain is a t -local domain can greatly simplify the proof that the domain in question is a valuation domain. But first let us explain the "t-ideal" terminology, that might be alien to some.

Let D be an integral domain with quotient field K, let $F(D)$ be the set of non-zero fractional ideals of D and for $A \in F(D)$, let $A^{-1} = \{x \in K : xA \subseteq D\}$. The functions on $F(D)$ defined by $A \mapsto A_v = (A^{-1})^{-1}$ and $A \mapsto A_t = \cup\{\mathbf{a}_v : 0 \neq \mathbf{a}$ is a finitely generated subideal of $A\}$, called the v- and t-operations, come under the umbrella of star operations discussed in sections 32 and 34 of [G], where the reader can find proofs of the statements made here about the v- and t-operations. An ideal A is a (v-) t-ideal if $A = (A_v)\ A_t$ and a (v-) t-ideal of finite type if $A = B_v$ for some finitely generated $B \in F(D)$. Next the t-operation is a star operation of finite type in the sense that $A \in F(D)$ is a t-ideal if and only if for each finitely generated nonzero subideal I of A we have $I_v \subseteq A$. An integral ideal maximal w.r.t being an integral t-ideal is called a maximal t-ideal and is a prime ideal. Finally every t-ideal is contained in a maximal t-ideal. Any unexplained terminology is straightforward and well accepted and usually comes from [K] or [G].

Proposition A. If D is a quasi local domain and the maximal ideal of D is minimal over (i.e. is a radical of) an integral t-ideal then D is t-local.

Proof. The proof can be found, couched in star operations of finite character, in [HH, (5) of Prop. 1.1] or in [Z-HD].

Corollary AA. If D is a quasi local domain and the maximal ideal of D is minimal over (i.e. is the radical of) a principal ideal of D, then D is t-local.

Obvious because a principal ideal is a t-ideal.

Corollary AB. If D is a quasi local domain and the maximal ideal of D is principal then D is t-local.

Follows from Corollary AA.

Corollary AC. A one dimensional quasi local domain is t-local.

Follows from the fact that in this case the maximal ideal is a minimal prime over every ideal contained in it.

Proposition AD. If (D, M) is quasi local and for every pair of prime ideals P, Q of D, we have $P \subseteq Q$ or $Q \subseteq P$, i.e. spec(D) is treed, or linearly ordered under inclusion, then D is t-local.

Let $I = (x_1, x_2, ...x_n) \subset M$ be a nonzero ideal of D and let P be a minimal prime of I. Then spec(D) being treed forces P to be unique. Now let, for each $i = 1, 2, ..., n$, $P(x_i)$ be the minimal prime of x_i. Again by the linearity of order of spec(D), for some $1 \leq k \leq n$, $P(x_k) \supseteq P(x_j)$ for $j \neq k$. So $P(x_k) \supseteq I$ and so $P(x_k) \supseteq P$. But as $x_k \in P$, $P(x_k) \subseteq P$. Whence every proper nonzero finitely generated ideal of D is contained in a prime ideal of D that is minimal over a principal ideal and hence is a t-ideal, by Proposition A, which is P in this case. Thus $I_v \subseteq P \subseteq M$. Since I is arbitrary as a finitely generated ideal, M is a t-ideal.

A nonzero element $c \in D$ is called comparable in D if for all $x \in D$ we have $(c) \subseteq (x)$ or $(x) \subseteq (c)$.

These elements were introduced and studied in [AZ] to prove a Kaplansky type theorem: An integral domain D is a valuation domain if and only if every nonzero prime ideal of D contains a comparable element. An important part of the result was the proof of the fact that the set of all comparable elements of D is a saturated multiplicative set.

Of course D is a valuation domain if and only if every nonzero element of D is comparable and this was used in [GMZ] to show that a GCD domain D is a valuation domain if and only if D contains a non unit comparable element. But there was more in store for us. In [GMZ] a part of the following observation was proved.

Proposition B (cf [GMZ, Theorem 2.5]). An integral domain D that contains a non unit comparable element is a t-local domain while a t-local domain may not contain a comparable element.

Proof. Let D be an integral domain and let d be a non unit comparable element in D. We first show that D is quasi local. Suppose by way of contradiction that there exist two co-maximal non unit elements x, y in D, i.e. $rx + sy = 1$ for some $r, s \in D$. Now as d is comparable $d|rx$ or $rx|d$. So rx has a non unit comparable factor d or, being a factor of d, rx is non unit comparable element. Thus rx has a non unit comparable factor h. Similarly sy has a non unit factor k. Since h, k are comparable, $h|k$ or $k|h$, say $h|k$. Thus assuming that $rx + sy = 1$ we get the contradictory conclusion that a non unit divides a unit. So, D is quasi local, with say maximal ideal M. Next let $x_1, x_2, ..., x_n \in M$ and note that as above, each of the x_i has a non unit comparable factor h_i. Thus $(x_1, x_2, ...x_n) \subseteq (h_1, h_2, ...h_n)$. Next since h_1, h_2 have each a non unit common factor k_1 ($= h_1$ or h_2). So, $(x_1, x_2, ...x_n) \subseteq (h_1, h_2, ...h_n) \subseteq (k_1, h_3...h_n)$. Continuing this process we eventually get a non unit comparable element k such that $(x_1, x_2, ...x_n) \subseteq (h_1, h_2, ...h_n) \subseteq (k)$. Thus $(x_1, x_2, ...x_n) \subseteq (k) \subseteq M$. But as $(x_1, x_2, ...x_n) \subseteq (k)$ implies $(x_1, x_2, ...x_n)_v \subseteq (k)$ we conclude that for each finitely generated ideal $(x_1, x_2, ...x_n) \subseteq M$, $(x_1, x_2, ...x_n)_v \subseteq M$. Thus D is a t-local domain. For the converse note that a one dimensional quasi local domain has only one nonzero prime ideal and so is a valuation ring if and only if it contains a non unit comparable element, by the Kaplansky type theorem mentioned above. The proof is complete once we note that there do exist one-dimensional, Noetherian quasi local domains that are not valuation domains.

A fractional ideal $I \in F(D)$ is said to be (v-) t-invertible if there is $J \in F(D)$

such that $((IJ)_v = D)$ $(IJ)_t = D$. A domain D is a Prufer v-multiplication domain, PVMD, if every finitely generated $I \in F(D)$ is t-invertible. It is well known (see Griffin [Gr]) that D is a PVMD if and only if D_M is a valuation domain. Obviously every invertible ideal is t-invertible. Note that a GCD domain D is a PVMD, because for each finitely generated nonzero ideal I of D we have I_v principal.

Corollary BA. A PVMD D is a valuation domain if and only if D contains a non unit comparable element.

Follows from the fact that a t-local PVMD is a valuation domain anyway and a valuation domain that is not a field must contain many non unit comparable elements.

This corollary is more interesting in that a GCD domain is a PVMD. Now here comes something a tad surprising. Call an integral domain D atomic if every nonzero nonunit of D is expressible as a finite product irreducible elements.

Corollary BB. An atomic domain that contains a non unit comparable element is a DVR.

Proof. Let D be an atomic domain and let d be a nonunit comparable element in D. Then by Proposition B, D is t-local with maximal ideal M. Let h be an irreducible factor of d. Then h is a comparable element, being a factor of a comparable element. So, for every x in D, $h|x$ or $x|h$. Now as h is irreducible $x|h$ means that x is a unit or $x = h$. Thus for all non units $x \in D$, $h|x$. That is $M = hD$. But then h is a prime. Next, as for each non unit $x \in D\backslash\{0\}$ $h|x$ we have $x = x_1h$ and if x_1 is a nonunit then $x_1 = x_2h$ and so $x = h^2x_2$. Continuing this way we can get $x = h^r x_r$. Because D is atomic, for each non unit $x \in D\backslash\{0\}$ there is $n = n(x)$ such that $x = h^n x_n$ where x_n is a unit. But then D is a DVR.

Remark BC. I had proved Corollary BB for Noetherian domains. Seeing that Tiberiu Dumitrescu suggested the atomic domain assumption. With hindsight we can prove the following result.

Corollary BD. Let D be a domain that contains a non-unit comparable element. Then D contains an atom a if and only if a is the generator of the maximal ideal of D and hence a comparable element.

Proof. Indeed D is t-local with maximal ideal D, by Proposition B. Let h be a nonunit comparable element of D. Then $h|a$ or $a|h$. If $h|a$ then as a is an atom and h a non-unit, h and a must be associates, so a is a comparable element. If, on the other hand, $a|h$ then a is comparable, being a factor of a comparable element. Thus as above $aD = M$. The converse is obvious, indeed if the maximal ideal M of a local domain D is principal and $M = Da$ then, up to associates, a is the only atom in D.

But the presence of a non unit comparable element in a domain D does more to the domain than just show that D is a t-local domain, as shown in [GMZ, Theorem 2.3]. We restate it and suggest that for the proof the readers look up [GMZ] here:

Proposition C. ([GMZ, Theorem 2.3]). Suppose the integral domain D contains a nonzero non-unit comparable element; let Y be the set of nonzero comparable elements of D. Then:

(1) $P = \cap\{(c) : c \in Y\}$ is a prime ideal of D and $D\backslash P = Y$.

(2) D/P is a valuation domain.

(3) $P = PD_P$.

(4) D is quasi local, P is a comparable ideal of D, and dim $D =$ dim (D/P) + dim (D_P)

Moreover, if J is any integral domain such that there is a non maximal prime ideal Q of J such that (a) J/Q is a valuation domain, and (b) $Q = QJ_Q$, then each element of $J\backslash Q$ is comparable. If, in addition, Q is minimal with respect to properties (a) and (b), then $J\backslash Q$ is the set of nonzero comparable elements of J. (Here an ideal I being comparable means that I compares with every other ideal under inclusion.)

Corollary CA. Suppose D contains a non-unit comparable element; let Y be the set of all comparable elements of D. D is a valuation domain if and only if $\cap\{(c) : c \in Y\} = 0$.

Follows from (1) and (2) of Proposition C.

Corollary CA. If a domain D contains a non unit comparable element then the maximal ideal of D is generated by some non unit comparable elements.

Obvious.

Note that if p is a prime element of a domain D then for each x in D, $(p) \cap (x) = (x)$ or $(p) \cap (x) = (px)$. So, $(p, x)^{-1} = \frac{(p)\cap(x)}{px} = (\frac{1}{p})$ or (1). But then $(p, x)_v = p$ or (1). So, if a prime element p belongs to a maximal t-ideal M then $M = (p)$. So, if a prime element p belongs to a t-local ring (D, M) then $M = pD$ consequently p is a comparable element of D. It is well known that if p is a prime element in an integral domain then $\cap(p^n)$ is a prime ideal (See e.g. Kaplansky [Kap, Exercise (5), page 7].

Proposition D. If a domain D contains a non-unit comparable element c then for every non-unit comparable element x, we have that $\cap(x^n) = Q$ is a prime ideal such that D/Q is a valuation domain and $Q = QD_Q$.Conversely, if there is an element x in a domain D such that $\cap(x^n) = Q$ is a prime ideal such that D/Q is a valuation domain and $Q = QD_Q$, then D is t-local and x is a comparable element of D.

Proof. Indeed Q is an ideal, being an intersection of ideals. Now consider $S = D\backslash Q$ and let $a, b \in S$. Then $a \notin (x^m)$ for some positive integer m and $b \notin (x^n)$ for some positive integer n. Since x and hence x^m, x^n are comparable we conclude that $(a) \supsetneq (x^m)$ and $(b) \supsetneq (x^n)$. Now $(ab) \supsetneq (bx^m)$ and $(bx^m) \supsetneq (x^{n+m})$ which gives $(ab) \supsetneq (x^{n+m})$ meaning $ab \in S$ and Q is a prime.

From the above proof it follows that S consists of factors of powers of the comparable element x and so every element of S is comparable; this means D/Q is a valuation domain. Next let $a/t \in QD_Q$ where $a \in Q$ and $t \in D\backslash Q$. But then t divides some power of x and so $(a) \subsetneq (t)$ which means that for some non unit y we have $a = ty$. As $t \notin Q$, $y \in Q$. So $a/t = y \in Q$. Thus $QD_Q \subseteq Q$. The converse follows from Theorem 2.3 of [GMZ].

Indeed there are integral domains that may or may not be quasi local but have elements x such that $\cap(x^n) = Q$ is a prime ideal such that $Q = QD_Q$, but D/Q is not a valuation domain. Here are some examples using the $D+M$ construction of Gilmer that goes as: Let V be a valuation domain expressible as $V = k + M$ where k is a subfield of V and M is the maximal ideal of V and let D be a subring of k. The ring $R = D + M$ is called the $D + M$ construction (see [BG] and has some interesting properties due to the mode of this construction, as indicated in [BG]. Our model for these examples would be $V = k[[X]] = k + Xk[[X]]$ and D a subring of k, giving $R = D + Xk[[X]]$.

Example DA. Let D be a one dimensional quasi local domain with quotient field l contained in k and suppose that D is not a valuation domain. Then $R = D+Xk[[X]]$ is such that for each nonzero non unit x in D we have $\cap(x^n) = Xk[[X]]$ (obvious) and $Xk[[X]] = Xk[[X]]R_{Xk[[X]]} = Xk[[X]](l + Xk[X]])$. But $R/Xk[[X]] = D$.

What makes the above example work is the fact that for a non unit x in a one dimensional quasi local domain D we have $\cap(x^n) = (0)$. Call an integral domain D an Archimedean domain if for all non unit elements x in D we have $\cap(x^n) = (0)$.

Example DB. Let D be an archimeden domain with quotient field l contained in k and suppose that D is not a valuation domain. Then $R = D + Xk[[X]]$ is such that for each nonzero non unit x in D we have $\cap(x^n) = Xk[[X]]$ (obvious) and $Xk[[X]] = Xk[[X]]R_{Xk[[X]]} = Xk[[X]](l + Xk[X]])$. But $R/Xk[[X]] = D$.

Example DC. Following the construction $R = D + XD_S[X]$ of [CMZ], if s is a non unit element in S such that $\cap(s^nD) = (0)$ then $\cap(s^nR) = XD_S[X]$ a prime ideal, but $R/XD_S[X] = D$ may not be a valuation domain.

From t-local domains to valuation domains

Because in a valuation domain (V, M) every finitely generated ideal is principal, the maximal ideal M is obviously a t-ideal. So t-local domains are cousins of valuation domains, but, sort of far removed. For example, $R = Z_{(p)} + (X, Y, Z)Q[[X, Y, Z]]$, with $M = pZ_{(p)} + (X, Y, Z)Q[[X, Y, Z]]$ is obviously t-local, but $R[1/p] = Q[[X, Y, Z]]$ which is quasi local, but as far away from being t-local as it gets. On the other hand quotient rings of a valuation domain are valuation domains. So it is legitimate to ask: Under what conditions is a t-local domain a valuation domain?

Here we address this question. The following is a simple result that hinges on the fact that if A is a finitely generated ideal in a t-ideal I then $A_v \subseteq I$.

Proposition E. For a set of elements $x_1, x_2, ..., x_n$, in a t-local domain (D, M), the following are equivalent.

(1) $(x_1, x_2, ..., x_n)_v = D$.

(2) At least one x_i is a unit.

(3) $(x_1, x_2, ..., x_n) = D$.

Proposition F. The following are equivalent for a t-local domain (D, M).

(1) D is a valuation domain

(2) D is a GCD domain.

(3) D is a PVMD.

Proof. That (1) $\Rightarrow$ (2) $\Rightarrow$ (3) is straight forward. For (3) $\Rightarrow$ (1) note that in a PVMD every nonzero finitely generated ideal $(x_1, x_2, ..., x_n)$ is t-invertible. But by Proposition 1.12 of [ACZ], $(x_1, x_2, ..., x_n)$ is principal.

It is well known that a commutative integral domain D is coherent if and only if the intersection of every pair of finitely generated ideals is finitely generated. Call a domain D a finite conductor domain if the intersection of every pair of principal ideals of D is finitely generated. Indeed a finite conductor (FC) domain is a generalization of coherent domains. This name (FC domain) was used in [Z-FC] first.

Corollary FA. For an integrally closed t-local domain the following are equivalent.

(1) D is a valuation domain.

(2) D is a coherent domain.

(3) D is a finite conductor domain.

Here (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are all straightforward. For (3) $\Rightarrow$ (1) note that an integrally closed FC domain is a PVMD [Z-FC] and a t-local PVMD is a valuation domain.

Corollary FB. (Theorem 1 [Mc]). Let D be an integrally closed quasi-local domain whose primes

are linearly ordered by inclusion. Suppose that the intersection of any two principal ideals is finitely generated. Then D is a valuation domain.

Proof. By Proposition AD, D is t-local and by [Z-FC, Theorem 2], D is a PVMD. (Once it is established that D is t-local the argument used in Lemma 5 of [Z-FC] may be used.)

Call a nonzero element r, of a domain D, primal if for all $x, y \in D \backslash \{0\}$ $r|xy$ implies that $r = st$ where $s|x$ and $t|y$. A domain whose nonzero elements are all primal is called pre-Schreier. An integrally closed pre-Schreier domain was called Schreier by P.M. Cohn in his paper [C]. There he showed that a GCD domains is a Schreier domain.

A module M is said to be locally cyclic if every finitely generated submodule of M is contained in a cyclic submodule of M. Thus an ideal I of D is locally cyclic if for any finite set of elements $x_1, x_2, ...x_n \in I$ there is an element $d \in I$ such that $d|x_i$. Based on considerations initiated by McAdam and Rush [McR], the following result was proved: An integral domain D is pre-Schreier if and only if for all $a, b \in D \backslash (0)$ and $x_1, x_2, .., x_n \in (a) \cap (b)$ there is $d \in (a) \cap (b)$ such that $d|x_i$. Based on this we can make the following note.

Note FC. We show, following [Z-PS], that if D is a pre-Schreier domain and $a, b \in D \backslash (0)$, then the following are equivalent:

(1) $(a) \cap (b)$ is principal, (2) $(a) \cap (b)$ is finitely generated, (3) $(a) \cap (b)$ is a v-ideal of finite type.

Proof. Indeed (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are all straightforward. All we need is show (3) $\Rightarrow$ (1). For this note that if $(a) \cap (b) = (x_1, x_2, ...x_n)_v$, then,

$x_1, x_2, ...x_n \in (a) \cap (b)$. Since D is pre-Schreier, there is a $d \in (a) \cap (b)$ such that $d|x_i$. That is $(x_1, x_2, ...x_n) \subseteq (d)$. But then $(x_1, x_2, ...x_n)_v \subseteq (d)$. This gives $(d) \subseteq (a) \cap (b) = (x_1, x_2, ...x_n)_v \subseteq (d)$.

Call a domain D a v-finite conductor (v-FC) domain if for each pair $0 \neq a, b \in D$, $(a) \cap (b)$ is a v-ideal of finite type. Then from Note FC we can conclude that: A domain D is a GCD domain if and only if D is a pre-Schreier and a v-FC domain. With this preparation we have the following result.

Corollary FD. For a pre-Schreier t-local domain D, the following are equivalent:

(1) D is a valuation domain,
(2) D is a coherent domain,
(3) D is an FC domain,
(4) D is a v-FC domain,
(5) D is a GCD domain.

Indeed the above are not the only situations in which a domain becomes a valuation domain.

Proposition G. Suppose that D contains a non unit comparable element x and let $P = \cap(x^n)$. Then D is a valuation domain if and only if D_P is a valuation domain.

Proof. Indeed if D is a valuation domain, then, P is a prime and, so D_P is a valuation domain and so we have only to take care of its converse. The presence of a non unit comparable element makes D a t-local domain. Let's split the proper finitely generated ideals into two types: (a) ones that contain a non unit factor of a power of x and (b) ones that do not contain a non unit factor of a power of x. Ones in part (a) are principal by Theorem 2.4 of [GMZ] and ones in part (b) are principal proper ideals of D_P and hence are in PD_P. By Proposition D above, $PD_P = P$, so for each x in P, xD_P is an ideal of D. Now let $x_1, x_2, ...x_n \in P$ and consider $(x_1, x_2, ...x_n)$. Since D_P is a valuation domain $(x_1, x_2, ...x_n)D_P = dD_P$ and we can assume that x_i, d are in D. So for some $r_i \in D$ and $s_i \in D \backslash P$ we have $x_i = \frac{r_i}{s_i} d$. (As $d \in P$, $s_i | d$, the right hand side is in D). So $(x_1, x_2, ...x_n) = (\frac{r_1}{s_1} d, \frac{r_2}{s_2} d, ..., \frac{r_n}{s_n} d)$. Removing the denominators we get $s(x_1, x_2, ...x_n) = (t_1 d, t_2 d, ..., t_n d)$ or $s(x_1, x_2, ...x_n) = (t_1, t_2, ..., t_n)d$. As $s(x_1, x_2, ...x_n)D_P = (x_1, x_2, ...x_n)D_P = dD_P = (t_1, t_2, ..., t_n)dD_P$ we conclude that $(t_1, t_2, ..., t_n)D_P = D_P$. But that means that at least one of the t_i is in $D \backslash P$ and hence is a comparable element. But then, by Theorem 2.4 of [GMZ], $(t_1, t_2, ..., t_n)$ is principal generated by a comparable element t. Thus $s(x_1, x_2, ...x_n) = t(d)$. Since s and t are comparable we have two possibilities: (α) $u(x_1, x_2, ...x_n) = (d)$ or (β) $(x_1, x_2, ...x_n) = v(d)$. In both cases $(x_1, x_2, ...x_n)$ turns out to be a principal ideal of D. (In case (α) because $u|d$ in D.)

Applications.

We have already pointed out that Theorem 1, of [Mc] falls to the observation that a quasi local domain with treed spectrum is actually t-local (Corollary FB) and necessarily quasi local. A domain D a treed domain if Spec(D) is treed i.e. Spec(D) is a tree as a poset. Indeed Spec(D) is treed if and only if any two incomparable primes of D are co-maximal. Indeed if D is such that Spec(D) is treed then Spec(D_P) is treed for every nonzero prime ideal P of D. So, by Proposition AD, every nonzero prime ideal of D is a t-ideal. In particular, every maximal ideal of D is a t-ideal. Indeed as a general t-local domain D may not have Spec(D) treed, as the example at the start of the previous section indicates. So the class of domains with treed spectra is strictly contained in the class of domains whose maximal ideals are t-ideals. But in the presence of some extra conditions this distinction may disappear.

Proposition H. For a Prufer v-multiplication domain D, the following are equivalent.

(1) Every maximal ideal of D is a t-ideal

(2) Spec(D) is treed

(3) D is a Prufer domain.

Proof. (1) $\Rightarrow$ (3) For every prime t-ideal P of a PVMD D, we have D_P a valuation domain [MZ, Corollary 4.3] and if D_P is a valuation domain for every maximal ideal of D then D is well known to be a Prufer domain. (3) $\Rightarrow$ (2) is clear because in a Prufer domain D, D_P is a valuation domain for every nonzero prime ideal P and Spec (D_P) is treed. Finally (2) $\Rightarrow$ (1) has been explained above.

Indeed as an integrally closed finite conductor domain is a PVMD [MZ, Corollary 4.3] and a Prufer domain is finite conductor, and this leads to the following result.

Corollary HA. An integrally closed treed domain D is Prufer if and only if D is finite conductor.

Indeed, it is worth noting that a nonzero ideal I in an integral domain D is said to be of grade one if $I \neq D$ and I does not contain a set of elements forming a regular sequence of length ≥ 2. So, every t-ideal is a grade one ideal and every nonzero prime ideal in a treed domain is a grade one ideal.

For the next application we need to prepare a little. Let R be a regular local ring, with quotient field F and, with dim $R = n$, and let $m = (x_1, ..., x_n)R$ be the maximal ideal of R. Choose $i \in \{1, ..., n\}$, and consider the overring $R[x_1/x_i, ..., x_n/x_i]$of R. Choose any prime ideal P of $R[x_1/x_i, ..., x_n/x_i]$ such that $P \supseteq m$. The ring $R_1 = R[x_1/x_i, ..., x_n/x_i]_P$ is a local quadratic transform (LQT) of R, and, again, a regular local ring with dim $R_1 \leq n$. If we iterate the process we obtain a sequence $R = R_0 \subseteq R_1 \subseteq R_2 \subseteq ...$ of regular local overrings of R such that for each j, $R_j + 1$ is a LQT of R_j. After a finite number of iterations dim R_j is bound to stabilize, and the process of iterating LQTs of the same Krull dimension and ascending unions of the resulting sequences are of interest to algebraic geometers. For a description the reader may consult [Heinzer et al, HLOST] which got the author interested in the topic.

Let $R = R_0 \subseteq R_1 \subseteq R_2 \subseteq ...$ be a sequence of LQTs from a regular local ring. Of interest in recent papers such as {[Heinzer et al]} has been the ring $S = \cup_{j \geq 0} R_j$, dubbed in recent work as "Sannon's Quadratic Extension" to honor David Shannon [Sh] for his interesting contribution. Briefly, before Shannon, Abhyankar [Ab, Lemma 12] had shown that if the regular local ring R has dimension 2 then S is a valuation overring of R such that the maximal ideal m_S of S contains m. David Shannon, a student of Abhyankar's, [Sh, Examples 4.7 and 4.17] showed that if dim $R > 2$, S need not be a valuation ring.

Our purpose here is to look at S from a simple star-operation theoretic perspective, provide some direct straight-forward and brief proofs of some known results and point to known results that could simplify some of the considerations in recent work.

We start by gathering some information about S.

(1) $S = \cup_{j\geq 0} R_j$ as described above is a quasi local ring. Let m_S denote the maximal ideal of S. Then $m_S = \cup_{j\geq 0} m_j$ where m_j is the maximal ideal of the LQT R_j.

(2) S is integrally closed, as being integrally closed is a first order property which is preserved by directed unions and hence ascending unions.

S has another elementary property but that needs some introduction. Cohn [C], called an element r of an integral domain D primal if for all $x, y \in D$ $r|xy$ in D implies that $r = st$ where $s|x$ and $t|y$. He called an integrally closed D a Schreier domain if each nonzero element of D is primal and showed that a GCD domain (every pair of (nonzero) elements has a GCD) is Schreier. He also noted [C, page 255] that the property of being Schreier is a first order property. Now $S = \cup_{j\geq 0} R_j$ is an ascending and hence directed union of regular local rings and hence GCD domains. This gives us the next property of S.

(3) S is (at least) a Schreier domain.

Next, according to [HLOST, Proposition 3.8] there is an element $x \in m_S$ such that $m_S = \sqrt{xS}$. This gives us, in light of Proposition A, the property that is of interest to us, in this article.

(4) S is a t-local ring.

This is enough information to provide more satisfying statements and proof(s) of Theorem 6.2 of [Heinzer et al]

Theorem K. (cf [Heinzer et al, Theorem 6.2]) Let S be a quadratic Shannon extension of a regular local ring. Then the following are equivalent:

(1) S is a valuation domain

(2) S is coherent.

(3) S is a finite conductor domain.

(4) S is a GCD domain.

(5) S is a PVMD.

(6) S is a v-finite conductor domain.

Proof. The equivalence of (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) comes from Corollary FA. Now (1) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) follow from Proposition F, and as S is Schreier (1) $\Leftrightarrow$ (6) by Corollary FD.

Corollary KA. If S is not a valuation domain then S contains a pair of elements a, b such that $aS \cap bS$ is not a v-ideal of finite type.

This corollary is significant with reference to the proof of Theorem 6.2 of [Heinzer et al] in that there are PVMDs D,such as Krull domains, that contain elements a, b such that $aD \cap bD$ is not finitely generated, but is of finite type. Besides such an example is good to have.

From [HLOST, Proposition 4.1] we conclude that S has another property of interest.

(5) For each element $x \in m_S$ such that $m_S = \sqrt{xS}$, $T = S[1/x]$ is a regular local ring with $\dim(T) = \dim(S) - 1$.

So, if $\dim(S) = 2$ and m_S contains a nonzero comparable element then S is a valuation domain. Also if $\dim(S) > 3$ then S cannot be a valuation domain, whether S comains a comparable element or not. For a regular local ring T of $\dim(T) > 1$, T is not a valuation domain. Indeed if $m_S = pS$ is principal then, S is a non-valuation t-local domain that contains a comparable element, by Proposition G. Indeed Proposition G provides a definitive criterion that can be used to provide examples of non-valuation t-local domains containing a comparable element, even in dim 2. The examples are: (1) $D = Z_{(p)} + P$, where P is the maximal ideal in $R[[X]]$, Z is the ring of integers, R the field of real numbers and p a nonzero prime element in Z. Indeed $D_P = Q + XR[[X]]$ which is not a valuation domain. In the same vein, and this is suggested by Tiberiu Dumitrescu, we have (2) $D = Z_{(p)} + P$ where P is the maximal ideal of (X^2, X^3) of $Q[[X^2, X^3]]$. Here $D_P = Q[[X^2, X^3]]$ which is a well known one dimensional Noetherian domain that is not a valuation domain. While we are at it, let P be the maximal ideal of the n-dimensional regular local ring $Q[[X_1, X_2, ...X_n]]$. Then $D = Z_{(p)} + P$ contains a proper comparable element and and, of course, D_P is far from being a valuation domain. Finally, and it is related, a one dimensional domain that contains a nonzero non-unit comparable element is a valuation domain. This follows from the facts that: (1) The presence of a comparable element forces the domain to be one dimensional t-local and (2) A domain is a valuation domain if and only if every nonzero prime ideal contains a nonzero comparable element [AZ].

Call the saturation of the set $\{x^n : n \in N\}$, span of x and denote it by $Span(x)$.

(6) If $x \in m_S$ such that $m_S = \sqrt{xS}$ then (a) for every non unit h in $span(x)$ we have $m_S = \sqrt{hS}$ and (b) m_S is generated by non units in $span(x)$.

Bibliography

[Ab] S. Abhyankar, On the valuations centered in a local domain. Amer. J. Math. 78 (1956), 321–348.

[ACZ] D. D. Anderson, G.W. Chang and M. Zafrullah, Integral domains of finite t-character, 20 J. Algebra 396 (2013) 169–183.

[AZ] D. D. Anderson and M. Zafrullah, On a theorem of Kaplansky, Boll. Un. Mat. Ital. A(7), 8 (1994), 397-402.

[BG] E. Bastida and R. Gilmer, Overrings and divisorial idels of rings of the form $D + M$, Michigan Math. J. 20(1973) 79-95.

[C] P.M. Cohn, Bezout rings and their subrings, Proc. Cambridge Phil. Soc. 64 (1968) 251-264.

[CMZ].D.Costa, J.Mott and M.Zafrullah,The construction $D + XD_S[X]$, J.Algebra53(1978)423–439.

[G] R.Gilmer, Multiplicative Ideal Theory, Marcel Dekker, NewYork,1972.

[GMZ] R. Gilmer, J. Mott and M. Zafrullah, t-invertbility and comparability, in Commutative Ring Theory (Editors: P.-J. Cahen, D. Costa, M. Fontana and S.-E. Kabbaj), Marcel Dekker, New York, 1994, 141-150

[Gr] M. Griffin, Some results on v-multiplication rings, Canad. J. Math. 19(1967), 710-722.

[Heinzer et al] L. Guerrieri, W. Heinzer, B. Olberding and Matt Toeniskoetter, Directed unions of local quadratic transforms of regular local rings and pullbacks, Arxiv preprint.

[HH] J.R. Hedstrom and E. Houston, Some remarks on star-operations, J. Pure Appl. Algebra18(1980) 37–44.

[HLOST] W. Heinzer, K. A. Loper, B. Olberding, H. Schoutens and M. Toeniskoetter, Ideal theory of infinite directed unions of local quadratic transforms, J. Algebra 474 (2017), 213–239

[K]. I Kaplansky, Commutative Rings, Allyn and Bacon, 1970.

[Mc] S. McAdam, Two conductor Theorems, J. Algebra 23(1972) 239-240.

[McR] S. McAdam and D. Rush, S c h r e i e r r i n g s , Bull. London Math. Soc. 10 (19781, 77-80.

[MZ] J. Mott and M. Zafrullah, On Prufer v-multiplication domains, Manuscripta Math. 35 (1981),1-26.

[Sh] D. Shannon, Monoidal transforms of regular local rings. Amer. J. Math. 95 (1973), 294–320.

[Z-FC] M.Zafrullah, On finite conductor domains, Manuscripta Math.24(1978)191–203.

[Z-PS] M. Zafrullah, On a property of pre-Schreier domains, Comm. Algebra 15 (1987), 1895-1920.

[Z-HD] http://www.lohar.com/mithelpdesk/hd0504.pdf