QUESTION (HD 1702): Let $A \subseteq B$ be an extension of domains. What is the difference between "$A \subseteq B$ is G2-stable" and "B is t-linked over A"?

ANSWER: Let's see what these notions mean, before making a decision. Uda [Uda, G2] called an extension $A \subseteq B$ of domains G2-stable if every ideal F of pgrade ≥ 2 of A generates an ideal FB of pgrade ≥ 2 in B. (Here pgrade, or polynomial grade, is a sort of improvement on the notion of grade of an ideal. In the addendum below we will include some introduction to it.) With reference to the extension $A \subseteq B$, Dobbs, Houston, Lucas and Zafrullah [DHLZ] called B t-linked over A if for a finitely generated nonzero ideal F of A, $F^{-1} = A$ implies $(FB)^{-1} = B$.

Now, an ideal F being of pgrade ≥ 2 means , in Uda's words from [Uda,G2, page 48], that F contains a finitely generated ideal G such that $G^{-1} = A$. So, looking from this angle the two concepts are identical. The difference seems to be in the languages they were couched in. The language of grade is a bit off-beat and somewhat restrictive, while "t-linked" seems to directly generalize "d-linked" i.e. if F in A is such that $F = A$ then $FB = B$. In fact this was the analogy that was at the back of our minds when we introduced the notion. We actually proved an analogue of "D is a Prufer domain if and only if every overring of D is integrally closed" for PVMDs as "D is a PVMD if and only if every t-linked overing of D is integrally closed" (Theorem 2.10, [DHLZ]). Call it power of the language or the focus that none of the authors of [DHLZ], realized that the concept of t-linked was related to G2-stability.

As it stands, B being t-linked over A is equivalent to " For each ideal F of A, $F_t = A$ implies $(FB)_{t_B} = B$. (Here t_B denotes the t-operation of B.) This has in turn caused the terminology of "v-linked", "$*$-linked" "$*_1 - *_2$-linked" etc. So, perhaps, the language has won.

As far as I am concerned, I would rather stick to "t-linked". My reason: To be of pgrade ≥ 2 an ideal has to be proper, but there are situations, as in a discrete rank one valuation domain where an ideal being proper (nonzero) is equivalent to the ideal being of grade one. Looking from this angle consider $A = k[[X, Y]]$, a two dimensional regular local ring and $B = k[[X, Y]]_{(X)}$. Being a ring of fractions of A, B is t-linked over A. Now take the ideal $I = (X, Y)$ of A. Then I is of grade 2, and hence of pgrade ≥ 2 but we cannot consider the grade of IB, because $IB = B$. Note here that as we shall see below the concepts of grade and pgrade are both defined for proper ideals, they coincide for noetherian domains and the above example is noetherian.

Kaplansky [K] calls a sequence $a_1, ..., a_n$ of elements of D an R-sequence if $(a_1, ..., a_n) \neq D$ and a_i is not a zero divisor on the module $D/(a_1, ..., a_{i-1})$ for $i = 1, ..., n$. The classical grade of an ideal Iof D, denoted by $g(I)$, is then the length of a longest R-sequence of elements of I. On the other hand pgrade(I) $= \lim_{m \to \infty} \operatorname{grad} e(ID[t_1, ..., t_m])$. Indeed the concept of polynomial grade is involved. Yet it depends upon the definition of classical grade and so before considering pgrade of an ideal IR we must ascertain that $IR \neq R$.

While the concept of pgrade is involved, the following can be shown: If IR is generated by n elements and $IR \neq R$, then pgrade(IR) $\leq n$. Moreover

$\mathrm{pgrade}(IR) = \mathrm{grad}\, e(ID[t_1, ..., t_n]$.

The definition and the result can be gleaned from Northcott's book [N]. I picked these up from [HM]. Of course the needed (over) simplification is mine.

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