QUESTION (HD 1703): I was reading the paper of P. Cohn "Bezout rings and their subrings" and I'm stuck in the proof of proposition 2.7: if R is a Schreier ring, then $R[x]$ is again a Schreier ring.

There are two things that I don't understand well. The first one: which theorem of Kronecker does Cohn refer to when he writes "Kronecker's lemma"?. I don't have access to Jaffard's book, but I have a guess: let R be an integral domain and let f, g be polynomials in $R[x]$. If $f = \sum_{i=0}^{n} a_i x^i$ and $g = \sum_{j=0}^{m} b_j x^j$, then $a_i b_j$ is integral over the ideal A_{fg} for all $0 \leq i \leq n$ and $0 \leq j \leq m$. (Here A_f means the content-ideal of a polynomial, i.e., $A_f = (a_0, a_1, \ldots, a_n)$) Is my guess right?

My second question is about the last part of the proof. Is the Riesz interpolation property the same as the characterization of Pre-Schreier domains given in theorem 1.1 of the paper "On a property of Pre-Schreier domains"? If so, I don't follow why Cohn uses it on K instead of over R, or is that Riesz interpolation property is another thing and then we can't use it on R, but rather on K. In that case, I don't see where is it used the hypothesis that R is a Pre-Schreier domain.

Thanks in advance for your answer. (This question was asked by Xam Diaz.)

ANSWER: A copy of Cohn's paper is available at:

http://www.lohar.com/researchpdf/bezout_rings_and_their_subrings.pdf

and a copy of Jaffard's theorem that Cohn refers to can be found at the link:

http://www.lohar.com/Jaffard%20page%2099.pdf .

While Jaffard credits the theorem to Kronecker, he does not provide a reference to Kronecker in the bibliography of his book. In any case the line of proof indicates that he is using the fact that if A is an integrally closed domain with quotient field K, then for $f = \sum a_i X^i$, $g = \sum b_j X^j \in K[x]\backslash(0)$, $A_{fg}V = (A_f A_g)V$ for each valuation overring V of A, where A_f denotes the content, that is the ideal generated by the coefficients of f. Now A is integrally closed if and only if A is the intersection of its set of valuation overrings $F = \{V_\alpha\}_{\alpha \in I}$ i.e. $A = \cap_{\alpha \in I} V_\alpha$. Now let ω be the star operation defined on A, by $I \mapsto I_\omega = \cap_\alpha I V_\alpha$, then $(A_{fg})_\omega = \cap_\alpha A_{fg} V_\alpha = \cap_\alpha A_f A_g V_\alpha = (A_f Ag)_\omega$.

For a basic introduction to star operations, look up sections 32 and 34 of Gilmer's book: Multiplicative ideal theory, Marcel Dekker, 1972 or my paper: Putting t-invertibility to use, Non-Noetherian commutative ring theory, 429-457, Math. Appl., 520, Kluwer Acad. Publ., Dordrecht, 2000 (available on my web-page at http://www.lohar.com/mit.html

Now let $c \in A\backslash 0$ and $c \mid fg$ then $A_{fg} \subseteq cA$ and so $(A_{fg})_\omega \subseteq (cA)_\omega = cA$. But then $(A_f A_g)_\omega \subseteq cA$. That means c divides every element of $(A_f A_g)_\omega$ and hence in particular the elements $a_i b_j$.

The question, about Cohn's proof, that you have raised can be answered in the following two ways.

(1) (4) of Theorem 2.2 of Fuchs [Ann. Scoula Norm. Sup. P i s a 19 (19651, 1-34] provides a characterization of Riesz groups and Cohn based the definition of Schreier domains on that characterization. Once established that the group of divisibility of a Schreier domain is a Riesz group, it is perfectly legitimate to

use the defining property, the Riesz interpolation property, of Riesz groups. He is not reasoning in K, he is reasoning in K^*/U the group of divisibility of R. (By the way, in [Manuscripta Math. 80(1993), 225-238] I use (4) of Theorem 2.2 of Fuchs, mentioned above, to conduct a primal element based study of Riesz groups.)

(2) The question that you have raised, has been considered and completely answered in section 4 of my paper with Dan Anderson: The Schreier property and Gauss Lemma, Bollettino U. M. I. (8) 10-B (2007), 43-62. In that paper we (Dan and I) show that if A is Schreier then so is $A[X]$ in the spirit of Theorem 1.1 of my paper "On a property of pre-Schreier domains".

I hope these comments answer your query. But if you still have doubts or concerns, feel free to get back to me.

Xam Diaz came back with

"This is the first time that I see the so-called star operations. Right now, I'm writing some notes about Gauss' theorem: R is a UFD if only if $R[x]$ is a UFD. I want to use only elementary ring-theoretic methods that not go beyond basic commutative algebra, so I hope you can understand my concern about the use of star operations. On the other hand, I have in mind the following approach: it can be proved that if $f = \sum_{i=0}^{n} a_i x^i$ and $g = \sum_{j=0}^{m} b_j x^j$, then $a_i b_j$ is integral over the ideal A_{fg} for all $0 \leq i \leq n$ and $0 \leq j \leq m$. Now, since $c \mid fg$, then clearly $A_{fg} \subseteq (c)$, therefore $a_i b_j$ is integral over (c). From this it's easy to show that $\frac{a_i b_j}{c}$ is integral over R, but R is integrally closed, hence $c \mid a_i b_j$.

What do you think of my approach?, is it right?

Regarding my other question. I read your paper "The Schreier Property and Gauss' Lemma" and certainly I understand your approach using your characterization of Pre-Schreier domains given your paper "On a property of Pre-Schreier domains"."

Certainly I hadn't seen the result that Xam told me about and I could not see it directly, so I asked for a proof. (I do not usually delve in integral closures of ideals.) In the meantime I offered the following alternative to Cohn's proof, avoiding star operations, as follows.

Note the following two well known facts.

I. An integral domain A is integrally closed if and only if A is an intersection of valuation rings V_α between A and the quotient field K of A. (See e.g. (11.12) of Nagata's book "Local Rings", Interscience Publishers.)

II. Let $f, g \in K[X]$ (where $K = qf(A)$) be two nonzero polynomials with $deg(g) = m$. Then $A_f^{m+1} A_g = A_f^m A_{fg}$.Let $f, g \in K[X]$ (where $K = qf(A)$) be two nonzero polynomials with $deg(g) = m$. Then $A_f^{m+1} A_g = A_f^m A_{fg}$. (See e.g. Theorem 28.1 of Gilmer's book [Multiplicative ideal theory, Marcel Dekker, New York 1972.)

Using II we can prove that

III. If V is a valuation ring between A and K and if $f, g \in K[X] \backslash \{0\}$ are two nonzero polynomials, then $A_{fg} V = A_f V A_g V$.

Let m be the degree of g. Then by II, $A_f^{m+1} A_g = A_f^m A_{fg}$. So $(A_f^{m+1} A_g)V = (A_f^m A_{fg})V$, or $A_f^{m+1} V A_g V = A_f^m V A_{fg} V$.

Now as A_f, A_g and A_{fg} are all finitely generated and as over a valuation domain finitely generated ideals are principal A_fV, A_gV and $A_{fg}V$ are all principal. Cancelling A_f^m from both sides of $A_f^{m+1}VA_gV = A_f^mVA_{fg}V$ we get $A_fVA_gV = A_{fg}V$.

Let A be integrally closed. Then by I, there is a set $F = \{V_\alpha\}_{\alpha\in I}$ of valuation domains between A and K such that $A = \cap V_\alpha$.

Now let $c \in A\backslash 0$ and $c \mid fg$ then $A_{fg} \subseteq cA$ and for each valuation domain $V_\alpha \in F$ we have $A_{fg}V_\alpha \subseteq cV_\alpha$ and, by III $A_{fg}V_\alpha = A_fV_\alpha A_gV_\alpha = A_fA_gV_\alpha$.

Thus $c|fg$ implies that $A_fA_gV_\alpha \subseteq cV_\alpha$. But then $\cap_\alpha(A_fA_g)V_\alpha \subseteq \cap_\alpha cV_\alpha = c\cap_\alpha V_\alpha = cA$. Indeed as $A_fA_g \subseteq (A_fA_g)V_\alpha$ for each α we have $A_fA_g \subseteq \cap_\alpha(A_fA_g)V_\alpha \subseteq cA$ and as above we can conclude that $c|\ (\sum a_iX^i)(\sum b_jX^j)$ implies $c|a_ib_j$, over integrally closed A.

Xam's insistence on using the statement he had offered indicated that he knew its proof. So I asked for it. In came a longish proof with, a somewhat insufficient statement, one nice step and the rest a standard argument used in several proofs. Before I give Xam's proof some introduction is in order.

Definition A. Let I be an ideal in a ring R. An element $r \in R$ is said to be integral over I if there exist an integer n and elements $a_i \in I^i, i = 1, ..., n$,

such that $r^n + a_1r^{n-1} + a_2r^{n-2} + \cdots + a_{n-1}r + a_n = 0$.

The set $\overline{I}$ of all elements integral over I is again an ideal and is called the integral closure of I. The best source known to me on this topic is the Swanson-Huneke book: [SH Integral closure of ideals, rings and modules, LMS Lecture Notes Series 336, Cambridge University Press.]

The event of an element r being integral over an ideal I is characterized in [SH, Proposition 1.1.7] as:

Proposition B. Let R be a ring, not necessarily Noetherian. For any element $r \in R$ and ideal $I \subseteq R, r \in I$ if and only if there exists an integer n

such that $(I + (r))^n = I(I + (r))^{n-1}$.

Next comes the following corollary ([SH, Corollary 1.1.8].

Corollary C. (Determinantal trick, cf. Lemma 2.1.8) Let I be an ideal in R and $r \in R$. Then the following are equivalent:

(1) r is integral over I.

(2) There exists a finitely generated R-module M such that $rM \subseteq IM$ and such that whenever $aM = 0$ for some $a \in R$, then $r \in \sqrt{0} : a$.

Moreover, if I is finitely generated and contains a non-zerodivisor, r is integral over I if and only if there exists a finitely generated faithful R-module M such that $IM = (I + (r))M$.

This much information is enough to prove a somewhat improved version of Xam's result. Let me record here first the statement of Dedekind-Mertens Lemma that appears in Gilmer's book, with the assumption that S is a ring and R a subring of S and A_f the R module generated by coefficients of f:

II'. Let $f, g \in S[X]$ be two nonzero polynomials with $deg(g) = m$. Then $A_f^{m+1}A_g = A_f^mA_{fg}$.

I will, however stick to integral domains as when these concepts were being developed integral domain were the only medium.

Theorem D. Let S be an integral domain with R a subring of S and let $f = \sum_{i=0}^{n} a_i x^i, g = \sum_{j=0}^{m} b_j x^j \in S[x]$ be two nonzero polynomials, such that $fg \in R[x]$. Then $a_i b_j$ is integral over R, for $i = 0, 1, ..., n; j = 0, 1, ..., m$. Moreover if for some nonzero $c \in R$ we have $c|fg$ in $R[x]$ then, for each pair i, j, $\frac{a_i b_j}{c}$ is integral over R.

Proof. By the Dedekind-Mertens Lemma (as given in II' above), $A_f^{m+1} A_g = A_f^m A_{fg}$ where A_f, A_g, A_{fg} denote the contents of the respective polynomials. Writing this equation as $A_f^m A_f A_g = A_f^m A_{fg}$ and setting $M = A_f^m$, $I = A_{fg}$ we get $A_f A_g M = IM$. Now, for each i, j, $a_i b_j \in A_f A_g$. So, for each i, j; $a_i b_j M \subseteq IM \subseteq M$. Then by Theorem 12 of Kaplansky [K, Commutative rings, Allyn and Bacon, 1970], $a_i b_j$ is integral over R, for $i = 0, 1, ..., n; j = 0, 1, ..., m$.

Next note that $c|fg$ in $A[x]$ implies that $A_{fg} \subseteq cR$. So I/c is an ideal of R and thus $(I/c)M \subseteq M$. Now consider $a_i b_j M \subseteq IM$. Dividing both sides of the inequality by c we get $(a_i b_j/c)M \subseteq (I/c)M$ and by the above observation $(a_i b_j/c)M \subseteq M$ and again by Theorem 12 of [K] we conclude that $a_i b_j/c$ is integral over R.

Theorem E (Statement offered by Xam) Let A be an integral domain and let $f = \sum_{i=0}^{n} a_i x^i, g = \sum_{j=0}^{m} b_j x^j \in A[x]$ be two nonzero polynomials. Then $a_i b_j$ is integral over the ideal A_{fg}, for $i = 0, 1, ..., n; j = 0, 1, ..., m$. Consequently if $c|fg$ in $A[x]$ for some $c \in A \backslash (0)$ then $a_i b_j/c$ is integral over A for each pair i, j.

Proof. By the Dedekind-Mertens Lemma, $A_f^{m+1} A_g = A_f^m A_{fg}$ where A_f, A_g, A_{fg} denote the contents of the respective polynomials. Writing this equation as $A_f^m A_f A_g = A_f^m A_{fg}$ and setting $M = A_f^m$, $I = A_{fg}$ we get $A_f A_g M = IM$. Now, for each i, j, $a_i b_j \in A_f A_g$. So, for each i, j; $a_i b_j M \subseteq IM$. Thus by Theorem C ((2) $\Rightarrow$ (1)) $a_i b_i$ is integral over $I = A_{fg}$. This means there are a positive integer s and elements $h_l \in A_{fg}^l$ such that

$(a_i b_j)^s + h_1 (a_i b_j)^{s-1} + h_2 (a_i b_j)^{s-2} + ... + h_{s-1}(a_i b_j) + h_s = 0$..........(1)

Now $c|fg$ leads to $A_{fg} \subseteq (c)$ and to $h_l \in (A_{fg})^l \subseteq (c^l)$ meaning $c^l | h_l$ for each $l = 1, 2, .., s$. Dividing the above equation by c^s we get

$(a_i b_j/c)^s + (h_1/c)(a_i b_j/c)^{s-1} + h_2/c^2 (a_i b_j/c^{s-2})^{s-2} + ... + h_{s-1}/c^{s-1}(a_i b_j/c) + h_s/c^s = 0$, a polynomial equation in $a_i b_j/c$ with coefficients in A. Thus $a_i b_j/c$ is integral over A for each pair i, j.

Corollary E. Let A be an integrally closed integral domain with quotient field K and let $f = \sum_{i=0}^{n} a_i x^i, g = \sum_{j=0}^{m} b_j x^j \in A[x]$ be two nonzero polynomials. If for some nonzero $c \in A$ we have $c|fg$ in $A[x]$ then, for each pair i, j, $\frac{a_i b_j}{c} \in A$.

Proof. By Theorem D1 $\frac{a_i b_j}{c}$ is integral over A which is integrally closed and so $\frac{a_i b_j}{c} \in A$ for each pair i, j. (The same conclusion follows if we use Theorem D)

Corollary F. Let A be an integrally closed integral domain with quotient field K and let $f = \sum_{i=0}^{n} a_i x^i, g = \sum_{j=0}^{m} b_j x^j \in K[x]$ be two nonzero polynomials, such that $fg \in A[x]$. If for some nonzero $c \in A$ we have $c|fg$ in $A[x]$ then, for each pair i, j, $\frac{a_i b_j}{c} \in A$.

Proof A being integrally closed puts $a_i b_j$ in A. Now you can use either of the theorems to get the conclusion.

I was reluctant to heed Xam's suggestion that his statement had anything to do with Kronecker or Dedekind, as the notion of integral closure of an ideal is recent. But there does exist a theorem credited to Kronecker and the statement of the theorem is very close to what Xam said, but not quite it.

Theorem G. Let S be a commutative ring and let inside $S[X]$, $f(X) = \sum^{i} f_i X^i = g(X)h(X) = (\sum^{j} g_j X^j)(\sum^{k} h_k X^k)$. Then each $g_j h_k$ is integral over the subring R of S generated by the f_i's.

I have taken this statement, for the most part from Henri Lombardi's paper, "Hidden constructions in algebra (1): integral dependence" [J. Pure Appl. Algebra 167 (2002) 259–267]. In today's terms the theorem is an easy consequence of Kaplansky's Theorem 12 (in [K]) and as rings with zero divisors were not in vogue in Kronecker's day Theorem G is precisely Theorem D. But while a rudimentary form of what is known as Dedekind-Mertens was probably known to Kronecker and other algebraists of his time, it was given a sure footing in Dedekind's paper [D, Uber einen arithmetischen Satz von Gauß ¨ , Mittheilungen der Deutschen Mathematischen Gesellschaft in Prag, Tempsky, 1892, pp. 1–11], a year after Kronecker died. ? (If you are curious about the other name on Dedekind-Mertens lemma; Mertens, a student of Kronecker's, got his name attached to the lemma because he had offered a proof of the lemma with $m = (\deg\ g)^2$.) The point is that Kronecker didn't use the lemma and understandably his proof was involved.

Now let's see what Kronecker's statement actually looked like. Edwards records it on page 3 of his book: [Divisor Theory, Birkhauser, Boston-Basel-Berlin 1990]:

Theorem H. Let $a_0, a_1, ..., a_m; b_0, b_1, ...b_n$ be indeterminates and let R be the ring of polynomials in these indeterminates with integer coefficients. Let $c_0, c_1, ..., c_{m+n}$ be defined by $c_i = \sum_{\substack{j+k=i \\ 0 \le j \le m; 0 \le k \le n}} a_j b_k$. Then each of the $(m+1)(n+1)$ elements $a_j b_k$ are integral over the subring of R generated by $1, c_0, c_1, ..., c_{m+n}$.

Note that R here is a ring generated by $a_0, a_1, ..., a_m; b_0, b_1, ...b_n$ over the ring of integers and over this ring R Kronecker is considering the product of two polynomials with coefficients given as indeterminates, for distinction perhaps. You can see that it had to undergo a lot of "improvements" to get to the form of Theorem G.

Xam had also mentioned Dedekind's Prague theorem as a possible source of what he had in mind. As I do not have access to the actual statement of the theorem, I copy Edwards' statement below.

Theorem K (Dedekind's Prague Theorem). Let f and g be polynomials in one indeterminate whose coefficients are algebraic numbers. If all coefficients of fg are algebraic integers, then the product of any coefficient of f and any coefficient of g is an algebraic integer.

This result has no direct bearing on Xam's problem as such but it does get thrown around in connection with polynomials over the total quotient ring of a ring. On the other hand some authors (e.g. Edwards in the above cited book and Lombardi in the above cited paper) say that Dedekind's Prague theorem is a consequence of Kronecker's Theorem. If we look at the modern day statement of Kronecker's Theorem yes, because the ring of algebraic numbers is integrally closed. But if we go by the statement attributed by Edwards to Kronecker then he is talking about polynomials over a ring of polynomials over integers which not only does not cover algebraic numbers but also is an integrally closed ring and so is a direct consequence of its modern version, i.e. Theorem D. Incidentally the fact that Kronecker's theorem was about polynomials on a polynomial ring over integers is also supported by the statement of Kronecker's theorem given on page 103 of Lombardi and Quitte's book [Algèbre commutative Meéthodes constructives].

Professor Irena Swanson was exposed to a copy of this answer and so was Professor Henri Lombardi. I thank Irena for suggesting that a link to Cohn's paper be provided and I thank Henri for indicating that a version of Krnecker's Theorem can be found in his book Algebre commutative ... mentioned above.

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