QUESTION (HD 1704): Let D be a pre-Schreier domain. If D is an IDF domain, must $D[X]$ be an IDF domain? (Professor Frank Okoh put this question to me.)

ANSWER: Yes.

Explanation: A bit of introduction first. Let D be an integral domain with quotient field K. Call a nonzero element r in D primal if for all $x, y \in D \backslash \{0\}$, $r|xy$ implies $r = st$ where $s|x$ and $t|y$. Cohn [C] called an integrally closed integral domain D Schreier if each nonzero element of D is primal. A domain whose nonzero elements are primal was called pre-Schreier in [Z]. Note that in a pre-Schreier domain every irreducible element (atom) is prime. (In fact a primal atom in any domain, is prime. For let p be an irreducible element that is also primal and let $p|ab$. So $p = rs$ where $r|a$ and $s|b$, because p is primal. But as p is also an atom r is a unit or s is a unit. Whence $p|a$ or $p|b$.) An integral domain D is said to have the Irreducible Divisor Finite (IDF) property if every nonzero non unit of D is divisible by at most a finite number of non associated irreducible elements of D. Obviously, in a pre-Schreier domain the IDF property translates to: Every nonzero element is divisible by at most a finite number of non-associated primes.

Let's begin with recalling that a nonzer prime ideal P of the poynomial ring $D[X]$ with $P \cap D = (0)$ is called a prime upper to zero or just "upper to zero". Now if D is pre-Schreier, $D[X]$ may not be pre-Schreier, see e.g [Z, Remarks 4.6]. So, some irreducible elements of $D[X]$ may not be prime. However if f is an irreducible non-constant polynomial in $D[X]$ then f is primitive, i.e. the GCD of the coefficients of f is 1 and over a pre-Schreier domain a primitive polynomial is super-primitive [ZWB, Lemma 2.1], meaning $(A_f)_v = D$. ([ZWB, Lemma 2.1] was stated for Schreier domains but was proved using properties characterizing pre-Schreier domains only.) Now f being a non-constant polynomial, f must belong to an upper to zero P of $D[X]$ and because $(A_f)_v = D$ the upper to zero P, containing f, must be a maximal t-ideal [HZ, Theorem 1.4] (See [HZ] for an introduction to t-ideals, and note that a t-ideal maximal M among integral t-ideals is called a maximal t-ideal and it can be shown that a maximal t-ideal is a prime ideal.) Thus any prime upper to zero in $D[X]$ that contains an irreducible polynomial is a maximal t-ideal, when D is pre-Schreier.

Next, verifying the IDF property entails checking that each nonzero polynomial $g \in D[X]$ is divisible by at most a finite number of irreducible divisors. If g is constant then all the divisors of g come from D alone and there are finitely many irreducible divisors for each constant g. So, let g be non-constant. Obviously each irreducible divisor of g that comes from D is a divisor of each of the coefficients of g and so g has only finitely many irreducible divisors coming from D. Next, we can write $g = hf_1^{n_1} f_2^{n_2} ... f_r^{n_r}$ where $h \in K \backslash \{0\}$ and f_i are prime elements in $K[X]$. So g belongs to r height one prime ideals of $D[X]$ and of these say n are maximal t-ideals. Now let f be an irreducible divisor of g. As we have already established, every prime t-ideal containing f is a maximal t-ideal. Let $f = eh_1^{t_1} ... h_m^{t_m}$ where e is a unit in $K[X]$ and h_i are primes in $K[X]$ and $\partial(f) = \sum t_i \partial(h_i)$, where $\partial(f)$ denotes the degree of the polynomial f. Any irreducible polynomial f_1' belonging to the same m maximal t-ideals and the same degree as f would be of the form $f_1 = e_1 h_1^{s_1}...h_m^{s_m}$ where e_1 is a unit in $K[X]$ and $\partial(f) = \sum t_i \partial(h_i) = \sum s_i \partial(h_i) = \partial(f_1)$. As the h_i are fixed, there can only be a finite number of irreducible polynomials of the form $e_1 h_1^{s_1}...h_m^{s_m}$. Also if f_1 is irreducible in $D[X]$ then $e' f_1$, for $e' \in K$ (cannot be an irreducuible element in $D[X]$ unless e' is a unit in D. (For $e' f_1 = k$ and k irreducible $\Rightarrow (A_{e' f_1})_v = (A_k)_v = D \Rightarrow e'(A_{f1})_v = (A_k)_v = D$ or $e' D = D$.) Thus there can only be a finite number of non-associated irreducible elements of the same degree belonging to m maximal t-ideals as f. Now there are only finitely many degrees $\partial g - \partial f$ between ∂g and ∂f and we can argue as above to establish that for each degree there are at most a finite number of irreducible divisors of g. Now take any subset S of the set of the n maximal t-ideals. It may be that there is no irreducible divisor of g of any degree that belongs to all the members of S. Otherwise there would only be a finite number of irreducible divisors of g that belong to exactly all the members of S. Now there are less than 2^n such subsets and, each subset T can contribute only a finite number of irreducible divisors of g, coming from exactly the members of T.

The case of Schreier domains, i.e. integrally closed pre-Schreier domains, may be handled as follows: It is known that if D is Schreier then so is $D[X]$, according to [C]. So the non constant irreducible elements of $D[X]$ are prime and generators of uppers to zero containg them. Now D being IDF the constant irresucible divisors of a general non-constant $f \in D[X]$ come from D and so are finite, up to associates, and the non-constant irreducible divisor are finite, up to associates, because they are the generatorors of the uppers to zero containing them.

[C] P. Cohn, Bezout rings and their subrings, Proc. Cambridge Phil. Soc.64 (1968), 251-264.

[ZWB] M. Zafrullah, "Well behaved prime t-ideals" J . Pure Appl. Algebra 65(1990) 199-207.

[HZ] E. Houston and M. Zafrullah,"On t-invertibility II" Cornm. Algebra 17(8)(1989) 1955-1969 .

[Z] M. Zafrullah, On a property of pre-Schreier domains, Comm. Algebra 15 (1987), 1895-1920.

Notes: (1) Professor Tiberiu Dumitrescu, a very helpful HD supporter, after seeing the above, sent the following comment: Your proof yesterday shows that a PSP (primitive polynomials superprimitive) IDF domain has IDF polynomial extension.

Indeed it is an improvement in that there do exist PSP domains, i.e domains D such primitive polynomials over D are super primitive, that are not pre-Schreier. For one such example, look up Example 2.11 and the explanation after Proposition 3.2 in [Andeerson and Zafrullah, The Schreier property and gauss' Lemma, Bollettino U. MI, 8 (2007), 43-62].

(2) Professor Frank Okoh made suggestions towards improvement of this answer. All of them are gratefully acceptd and incorporated.

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